$t$-Deletion-$1$-Insertion-Burst Correcting Codes

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Abstract

Motivated by applications in DNA-based storage and communication systems, we study deletion and insertion errors simultaneously in a burst. In particular, we study a type of error named $t$-deletion-$1$-insertion-burst ($(t, 1)$-burst for short) proposed by Schoeny et al., which deletes $t$ consecutive symbols and inserts an arbitrary symbol at the same coordinate. We provide a sphere-packing upper bound on the size of binary codes that can correct $(t, 1)$-burst errors, showing that the redundancy of such codes is at least $log n + t - 1$. An explicit construction of a binary $(t, 1)$-burst correcting code with redundancy $log n + (t - 2) log log n + O(1)$ is given. In particular, we construct a binary $(3, 1)$-burst correcting code with redundancy at most $log n + 9$, which is optimal up to a constant.

Index Terms

DNA storage, error-correcting codes, deletions, insertions, burst error.

I. INTRODUCTION

DNA-based storage is a promising direction for future data storage due to its advantages such as storage density and durability. Due to the error behavior in DNA sequences, codes correcting deletions and insertions have recently attracted significant attention. Meanwhile, synchronization loss occurs due to timing uncertainty in communication and storage systems, which also leads to deletions and insertions.

The study of error-correcting codes against deletions and insertions dates back to 1965, when Levenshtein showed that the Varshamov-Tenengol’ts (VT) code can correct a single deletion. In the same work, Levenshtein proved the equivalence between $t$-deletion-correcting codes and $t$-insertion-correcting codes, and showed that the redundancy of a $t$-deletion-correcting code is asymptotically at least $t log n$. It was not until recent years that deletion correcting codes were reconsidered due to advances in DNA storage. The general problem of correcting $t$ arbitrary deletions was considered in a series of works. The state-of-the-art result of them is the one from, which constructed a $t$-deletion-correcting code with redundancy $4t log n + o(log n)$.

Both in DNA-based storage and communication system, the deletion and insertion errors tend to occur in bursts (i.e. consecutive errors). Therefore, it is of great significance to design codes capable of correcting bursts of deletions/insertions errors. A code that can correct exactly $t$ consecutive deletions is called a $t$-burst-deletion-correcting code, and a code that can correct at most $t$ consecutive deletions is called a $t^2$-burst-deletion-correcting code.

In 1970, Levenshtein constructed a $2^t$-burst-deletion-correcting code, and provided asymptotic bounds on $t$-burst-correcting codes indicating that the least redundancy to correct $t$ consecutive deletions is asymptotically $log n + t - 1$. Cheng et al. provided three constructions of $t$-burst-deletion-correcting codes, among which the least redundancy is $t log (2^t + 1)$. Schoeny et al. constructed a $t$-burst-deletion-correcting code by combining the VT code with constrained coding and utilizing the shifted VT (SVT) code, which has redundancy $log n + (t - 1) log log n + t - log t$. For $t^2$-burst-deletion-correcting codes, the construction of Schoeny et al. has redundancy $(t - 1) log n + (\binom{t}{2} - 1) log log n + \binom{t}{2} + log log t$. Gabrys et al. reduced the redundancy to $[log t] log n + (\frac{t(t+1)}{2} - 1) log log n + O(1)$. The current best result is from Lenz et al. with redundancy $log n + \frac{t(t+1)}{2} log log n + O(1)$. In addition, the model of a burst of $t$ non-consecutive deletions (some $s \leq t$ deletions occur within a block of size $t$) are considered in and .

In addition to deletion and insertion errors, substitution errors are the most widely considered type of errors in error-correcting codes. In current DNA storage technology, ultimately one needs to design error-correcting codes against a combination of deletion, insertion, and substitution errors, which is by now a difficult problem. In this paper, we consider a type of error proposed by Schoeny et al., known as $t$-deletion-$1$-insertion-burst ($(t, 1)$-burst for short), which deletes a burst of $t$ consecutive symbols and then inserts an arbitrary symbol at the same coordinate. Such errors can be also seen as deleting a burst of $t - 1$ consecutive symbols and then replacing at most one symbol next to it. A code that can correct this kind of error will be called a $t$-deletion-$1$-insertion-burst correcting code ($(t, 1)$-burst correcting code for short). Similarly, one can define $1$-deletion-$t$-insertion-burst correcting codes ($(1, t)$-burst correcting codes for short) and it is routine to prove the equivalence of $(t, 1)$-burst correcting codes and $(1, t)$-burst correcting codes. Besides their own interest, such codes have also been applied in the model of a burst of $t$ non-consecutive deletions.
The rest of the paper is organized as follows. In Section II, we give the definitions and some notations used throughout the paper and review some previous results that will be used in our constructions. In Section III, we prove the equivalence between t-deletion-1-insertion-burst correcting codes and 1-deletion-t-insertion-burst correcting codes. A sphere-packing upper bound on the size of t-deletion-1-insertion-burst correcting codes is given in Section IV which leads to a lower bound of the redundancy of t-deletion-1-insertion-burst correcting codes. In Section V we present a code capable of correcting t-deletion-1-insertion-burst with redundancy $\log n + (t - 2)\log \log n + O(1)$ for general $t$. In particular, we present an almost optimal 3-deletion-1-insertion-burst correcting code in Section VI. Lastly, Section VII concludes this paper.

II. PRELIMINARIES AND RELATED WORK

A. Notations and Definitions

Let $F_2^n$ be the set of binary sequences of length $n$. In this paper, a sequence $x \in F_2^n$ is denoted as either $x_1x_2x_3\ldots x_n$ or $(x_1, x_2, x_3, \ldots, x_n)$. For integers $i \leq j$, the length of an interval $[i, j] = \{i, i + 1, \ldots, j\}$ is defined as $j - i + 1$.

A run of $x = (x_1, x_2, \ldots, x_n)$ is a maximal consecutive subsequence of the same symbol. For binary sequences, a run is a maximal consecutive 0s or 1s. Let $R(x)$ denote the run sequence of $x$, and the $i$th coordinate of $R(x)$ is denoted as $R(x)_i$, known as the run index of $x_i$. Here $R(x)_i = t$ indicates that $x_i$ lies in the $t$th run of $x$, where $t$ is counted starting from zero. Let $Rsyn(x) = \sum_{1 \leq i \leq n} R(x)_i$ and let $r(x)$ be the total number of runs in $x$. For example, if $x = 1011110000$, then ‘11’, ‘0’, ‘111’, ‘0000’ are the four runs, $R(x) = 0012223333$, $Rsyn(x) = 19$, and $r(x) = 4$.

A t-deletion-1-insertion-burst $(t, 1)$-burst for short) error over $x$ is a type of error which deletes $t$ consecutive symbols from $x$ and inserts an arbitrary symbol at the same coordinate. That is, a $(t, 1)$-burst error over $x = (x_1, x_2, \ldots, x_n)$ starting at the $j$th coordinate, $1 \leq j \leq n - t + 1$, will result in $(x_1, \ldots, x_{j-1}, y, x_{j+t-1}, \ldots, x_n)$, where $y$ is the inserted symbol. Note that a $(t, 1)$-burst error can also be seen as a burst of $t - 1$ consecutive deletions followed by at most one substitution next to it. Similarly, a 1-deletion-t-insertion-burst $(1, t)$-burst for short) error deletes one symbol from $x$ and inserts a binary string of length $t$ at the same coordinate. Continuing with the example above for $x = 1011110000$, if a $(3, 1)$-burst error starts at the very beginning then one gets either 011110000 or 11110000, and if a $(1, 3)$-burst error starts at the very beginning then one gets 011y2y3101110000 where $y_1y_2y_3$ represents the inserted string.

The $(t, 1)$-burst ball of $x$ is the set of all possible sequences obtained by a $(t, 1)$-burst error over $x$, denoted as $B_{t,1}(x)$. Similarly we have the $(1, t)$-burst ball $B_{1,t}(x)$.

**Example 1:** Suppose $x = 101000111$, $t = 4$, then

$$B_{4,1}(x) = \{000111, 100111, 110111, 101111, 101011, 101001, 101000\}.$$

A code $C \subseteq F_2^n$ is called a $(t, 1)$-burst correcting code if it can correct $t$-deletion-1-insertion-burst errors. That is, for any two distinct codewords $c_1, c_2 \in C$, $B_{t,1}(c_1) \cap B_{1,t}(c_2) = \emptyset$. The redundancy of such a code is given by $n - \log |C|$.

B. Related Work

In this subsection we briefly review some useful results about codes correcting deletions or insertions. Define the VT syndrome as $VT(x) = \sum_{i=1}^n t_i x_i$.

For $a \in Z_{n+1}$, the Varshamov-Tenengol’ts (VT) code $[19]$

$$VT_a(n) = \{x \in F_2^n : VT(x) \equiv a \pmod {n+1}\}$$

can correct a single deletion. Levenshtein [11] proved that the optimal redundancy of $t$-deletion-correcting codes is between $t \log n$ and $2t \log n$, so the VT code is optimal for a single deletion.

In 1970, Levenshtein [12] provided a code that can correct a burst of at most 2 deletions as follows:

$$L_a(n) = \{x \in F_2^n : Rsyn(0x) \equiv a \pmod {2n}\},$$

where $a \in Z_{2n}$ and the function $Rsyn(\cdot)$ is applied on $0x$, the concatenation of a single 0 and the sequence $x$.

For the $t$-burst-deletion-correcting codes, Cheng et al. [3] proposed a framework which represents $x$ as a $t \times \frac{n}{t}$ array and constructs codes for every row respectively:

$$A_t(x) = \begin{bmatrix} x_1 & x_{t+1} & \cdots & x_{n-t+1} \\ x_2 & x_{t+2} & \cdots & x_{n-t+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_t & x_{2t} & \cdots & x_n \end{bmatrix}.$$

In their constructions the least redundancy is $t(\log(\frac{n}{t}) + 1))$. 

Schoeny et al. [13] followed the framework as in [3] to represent a sequence as the array above. In their construction the first row is encoded by a variation of VT code with additional run-length constraints, and the other rows apply a code called shifted VT code (SVT):

\[
SVT(n; a, b, P) = \left\{ x \in F_q^n : VT(x) \equiv a \pmod{P}, \sum_{i=1}^n x_i \equiv b \pmod{2} \right\}
\]

which can correct a single deletion given the additional knowledge of the location of the deleted symbol within an interval of \( P \) consecutive coordinates. By choosing \( P \) as \( \log(n/\varepsilon) + 2 \), the redundancy of their codes is at most \( \log n + (t-1) \log n + t \log t \).

In the same paper [13], Schoeny et al. constructed a (2, 1)-burst correcting code, which was then applied to correcting a burst of \( t \) non-consecutive deletions:

\[
C_{2,1}(n; a, b) = \left\{ x \in F_q^n : VT(x) \equiv a \pmod{2n - 1}, \sum_{i=1}^n x_i \equiv b \pmod{4} \right\}.
\]

**C. Our Contributions**

In this paper, we first prove the equivalence of \((t, 1)\)-burst correcting codes and \((1, t)\)-burst correcting codes and thus we can focus only on \((t, 1)\)-burst errors. Next we compute the size of the error ball for \((t, 1)\)-burst errors. Unlike most situations in analyzing deletions/insertions, for \((t, 1)\)-burst errors the size of the error ball is a constant independent of the center and thus it leads to a neat sphere-packing type upper bound, implying that the redundancy is at least \( \log n + t - 1 \). For \( t \geq 3 \), we construct codes with redundancy \( \log n + (t - 2) \log \log n + O(1) \). In particular, we construct a \((3, 1)\)-burst correcting code with redundancy at most \( \log n + 9 \) which is optimal up to a constant.

**III. Equivalence of \((t, 1)\)-burst and \((1, t)\)-burst**

In this section, we will prove the equivalence of \((t, 1)\)-burst correcting codes and \((1, t)\)-burst correcting codes. In [12], Levenshtein proved the equivalence of \( t \)-deletion-correcting codes and \( t \)-insertion-correcting codes. Schoeny et al. [13] proved the equivalence of codes against a burst of deletions and a burst of insertions. Following a similar idea, we have the following theorem.

**Theorem 1:** A code \( C \) is a \((t, 1)\)-burst correcting code if and only if it is a \((1, t)\)-burst correcting code.

**Proof:** We only prove the ‘only if’ part. The ‘if’ part can be proved analogously. Suppose \( C \) is a \((t, 1)\)-burst correcting code but is not a \((1, t)\)-burst correcting code. Then, there exist two distinct codewords \( x, y \in C \), such that \( B_{1,t}(x) \cap B_{1,t}(y) \) is nonempty and thus contains some \( z \in F_q^{2+t} \). Assume that \( z \) is obtained by deleting \( x_i \) and inserting \( (a_1, \ldots, a_t) \) at the \( i \)th coordinate of \( x \), and is also obtained by deleting \( y_j \) and inserting \( (b_1, \ldots, b_t) \) at the \( j \)th coordinate of \( y \). Without loss of generality, we assume \( i \leq j \). Then, \( z \) will have the following two representations:

\[
z = (x_1, \ldots, x_{i-1}, a_1, \ldots, a_t, x_{i+1}, \ldots, x_n),
\]

\[
z = (y_1, \ldots, y_{j-1}, b_1, \ldots, b_t, y_{j+1}, \ldots, y_n).
\]

1) If \( j \geq i + t \), i.e., the coordinates of deletion and insertions of \( y \) are disjoint with the coordinates of deletion and insertions of \( x \), then we will get:

\[
(x_1, \ldots, x_{i-1}) = (y_1, \ldots, y_{i-1}), (a_1, \ldots, a_t) = (y_i, \ldots, y_{i+t-1}),
\]

\[
(x_{i+1}, \ldots, x_{j-1}) = (y_{i+t}, \ldots, y_{j-1}), (x_{j-t+1}, \ldots, x_j) = (b_1, \ldots, b_t), (x_{j+1}, \ldots, x_n) = (y_{j+1}, \ldots, y_n)
\]

by comparing the two representations of \( z \). Therefore, if we delete \((x_{j-t+1}, \ldots, x_j)\) from \( x \) and insert \( y_j \) at this coordinate, then we get \((x_1, \ldots, x_{j-t}, y_j, x_{j+1}, \ldots, x_n) \). If we delete \((y_1, \ldots, y_{i+t-1})\) from \( y \) and insert \( x_i \) at this coordinate, then we get \((y_1, \ldots, y_{i-1}, x_i, y_{i+t}, \ldots, y_n)\). From the equations above, we have \((x_1, \ldots, x_{j-t}, y_j, x_{j+1}, \ldots, x_n) = (y_1, \ldots, y_{i-1}, x_i, y_{i+t}, \ldots, y_n)\). Thus, \( B_{1,t}(x) \cap B_{1,t}(y) \neq \emptyset \), which is a contradiction to the hypothesis that \( C \) is a \((t, 1)\)-burst correcting code.

2) If \( j \leq i + t - 1 \), then the coordinates of deletion and insertions of \( x \) and \( y \) will have some intersection. Similarly we have:

\[
(x_1, \ldots, x_{i-1}) = (y_1, \ldots, y_{i-1}), (x_{j+1}, \ldots, x_n) = (y_{j+1}, \ldots, y_n)
\]

In this case, we delete \((x_{j-t+1}, \ldots, x_j)\) from \( x \) and insert \( y_{j-t+1} \) at this coordinate, then we get \((x_1, \ldots, x_{j-t}, y_{j-t+1}, x_{j+1}, \ldots, x_n)\). For \( y \), if we delete \((y_{j-t+2}, \ldots, y_{j+t-1})\) and insert \( x_{j+1} \) at this coordinate, then we get \((y_1, \ldots, y_{j-t}, x_{j+1}, y_{j+2}, \ldots, y_n)\). Since \( j \leq i + t - 1 \), we have \((x_1, \ldots, x_{j-t}) = (y_1, \ldots, y_{j-t})\). Therefore, \((x_1, \ldots, x_{j-t}, y_{j-t+1}, x_{j+1}, \ldots, x_n) = (y_1, \ldots, y_{j-t+1}, x_{j+1}, y_{j+2}, \ldots, y_n)\). Thus, \( B_{1,t}(x) \cap B_{1,t}(y) \neq \emptyset \), which is again a contradiction to the hypothesis that \( C \) is a \((t, 1)\)-burst correcting code.

With the equivalence of \((t, 1)\)-burst correcting codes and \((1, t)\)-burst correcting codes, we will only consider \((t, 1)\)-burst in the following sections.
IV. AN UPPER BOUND ON THE CODE SIZE

There are several upper bounds on the size of burst-deletion-correcting codes. Levenshtein \[12\] provided an asymptotic upper bound on the size of \( t \)-burst-deletion-correcting codes: if \( C \subseteq \mathbb{F}_2^k \) is a \( t \)-burst-correcting code, then \( |C| \leq 2^{n-t+1}/n \), which implies that the redundancy is asymptotically at least \( \log n + t - 1 \). Schoeny et al. \[13\] constructed a hypergraph whose vertices are all sequences of \( \mathbb{F}_2^{n-t} \) and the hyperedges are \( t \)-burst-deletion balls of all sequences in \( \mathbb{F}_2^{n-t} \). In this way, the problem turns into analyzing the matching number of the hypergraph and they provided an explicit upper bound on the size of \( t \)-burst-deletion-correcting codes as \( |C| \leq (2^{n-t+1} - 2^t)/(n - 2t + 1) \).

In this section, we will give a sphere-packing type upper bound on the size of \((t, 1)\)-burst correcting codes. A sphere-packing type bound is not easy to obtain for most models concerning insertions and deletions, since the size of the corresponding error ball usually depends on the choice of the center. However, we derive an unexpected result that the size of a \((t, 1)\)-burst ball is in fact a constant independent of the center.

**Theorem 2:** For every \( x \in \mathbb{F}_2^n \), \( t \geq 1 \),

\[ |B_{t,1}(x)| = n - t + 2. \]

To prove this result, we start with a lemma which suggests that the size of \( |B_{t,1}(x)| \) can be partitioned into two parts.

Let \( B_{t,1}(x) \) denote the sequences obtained by a \((t, 1)\)-burst error over \( x \), where the first and last deleted symbols are the same whereas the inserted symbol is different from those two. In other words,

\[ B_{t,1}(x) = \bigcup_{i=1}^{n-t+1} \left\{ x_1 \ldots x_{i-1}x_i'x_{i+1} \ldots x_n \middle| x_i = x_{i+t-1} \not\equiv x_i' \right\} \]

**Lemma 1:** For \( x \in \mathbb{F}_2^n \), \( t \geq 2 \),

\[ |B_{t,1}(x)| = |B_{t-1,0}(x)| + |B_{t,1}^r(x)|. \]

**Proof:** For every \( y \in B_{t,1}(x) \setminus B_{t,1}^r(x) \), \( y \) is obtained from \( x \) via a \((t, 1)\)-burst in the following two ways:

- the first and last deleted symbols are different;
- the first and last deleted symbols and the inserted symbol are the same.

In either case, \( y \) can be obtained by deleting \( t - 1 \) consecutive symbols from \( x \) and thus \( y \in B_{t-1,0}(x) \).

Then it suffices to prove \( B_{t-1,0}(x) \cap B_{t,1}^r(x) = \emptyset \). Assume to the contrary that there exists \( x' \in B_{t-1,0}(x) \cap B_{t,1}^r(x) \). Then, there exist \( 1 \leq i < n - t + 2, 1 \leq j < n - t + 1 \), such that

\[ x' = x_1 \ldots x_{i-1}x_i'x_{i+1} \ldots x_n \in B_{t-1,0}(x) \]

\[ x' = x_1 \ldots x_{j-1}x_j'x_{j+1} \ldots x_n \in B_{t,1}^r(x) \]

According to the length of deletion, one of \( \{j, j + t - 1\} \) cannot be in \( \{i, i + 1, \ldots, i + t - 2\} \). WLOG, we assume \( j \notin \{i, i + 1, \ldots, i + t - 2\} \). If \( j > i + t - 2 \), we have \( x_j' = x_{j+t-1} \), which is a contradiction to the definition of \( B_{t,1}^r(x) \). If \( j < i \), we have \( x_j' = x_j \), again a contradiction.

Now, we need to calculate \( |B_{t-1,0}(x)| \) and \( |B_{t,1}^r(x)| \). In \[12\], Levenshtein calculated the size of a \( t \)-burst-deletion ball for \( x \in \mathbb{F}_2^n \):

\[ |B_{t,0}(x)| = 1 + \sum_{i=1}^{t} (r(A_t(x)_i) - 1), \tag{3} \]

where \( A_t(x)_i \) is the \( i \)th row of the array \( A_t(x) \). In the next lemma, we will compute the size of \( B_{t,1}^r(x) \).

**Lemma 2:** Let \( x \in \mathbb{F}_2^n \), \( t \geq 2 \), then

\[ |B_{t,1}^r(x)| = n - \sum_{i=1}^{t-1} r(A_{t-1}(x)_i). \]

**Proof:** Let \( y_1, y_2 \in B_{t,1}^r(x) \) where the \((t, 1)\)-bursts start at the \( j_1 \)th and \( j_2 \)th coordinate, respectively (WLOG, assume \( j_1 < j_2 \)). That is, \( y_1 = (x_1, \ldots, x_{j_1-1}, y_1, x_{j_1}, \ldots, x_n) \) and \( y_2 = (x_1, \ldots, x_{j_2-1}, y_2, x_{j_2+t}, \ldots, x_n) \). Note that since \( x_{j_1} = x_{j_1+t-1} \not\equiv y_1 \), then \( y_1 \) and \( y_2 \) have distinct symbols on their \( j_1 \)th coordinate and thus \( y_1 \neq y_2 \). Therefore, the size of \( |B_{t,1}^r(x)| \) is equal to the number of pairs \((x_i, x_{i+t-1})\), where \( x_i = x_{i+t-1} \). That is,

\[ |B_{t,1}^r(x)| = \left| \{(x_i, x_{i+t-1}) | x_i = x_{i+t-1}, 1 \leq i \leq n - t + 1\} \right|. \]

Write \( x \) as a \((t-1) \times \frac{n}{t-1}\) array:

\[
A_{t-1}(x) = \begin{bmatrix}
\text{s}\text{t}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\begin{array}{cccc}
1 & x_t & \cdots & x_{n-t+2} \\
x_2 & x_{t+1} & \cdots & x_{n-t+3} \\
\vdots & \vdots & \ddots & \vdots \\
x_{t-1} & x_{2t-2} & \cdots & x_n
\end{array}
\end{bmatrix}
\]
From the representation of $A_{t-1}(x)$, the number of such pairs is equal to the number of two consecutive equal symbols in all rows. The $i$th row contributes $\frac{n}{t-1} - r(A_{t-1}(x)_i)$ to the size of $|B'_t(x)|$. Therefore,

$$|B'_t(x)| = \sum_{i=1}^{t-1} \left( \frac{n}{t-1} - r(A_{t-1}(x)_i) \right) = n - \sum_{i=1}^{t-1} r(A_{t-1}(x)_i).$$

Now we are ready for the calculation of $|B_{t,1}(x)|$.

**Proof of Theorem 2**: When $t = 1$, $B_{t,1}(x)$ is exactly the Hamming ball of radius $1$ and is of size $n + 1$. When $t \geq 2$, combining Equation 3 (substituting $t$ as $t - 1$), Lemma 1 and Lemma 2, we have

$$|B_{t,1}(x)| = |B_{t-1,0}(x)| + |B'_{t,1}(x)| = 1 + \sum_{i=1}^{t-1} (r(A_{t-1}(x)_i) - 1) + n - \sum_{i=1}^{t-1} r(A_{t-1}(x)_i) = n - t + 2.$$

**Example 2**: Let $n = 12, x = 10101100100, t = 4$, then

$$A_5(x) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

$$B_{3,0}(x) = \{011100100, 111100100, 101100100, 101001000, 101010100, 101011100, 101011110\},$$

$$B'_{4,1}(x) = \{100100100, 101011000, 101011110, 101011110\}.$$

Hence, $|B_{3,0}(x)| + |B'_{4,1}(x)| = 10 = 12 - 4 + 2$.

Theorem 2 shows that the size of the $(t,1)$-burst ball is independent of its center, which is rather rare in other models concerning deletions or insertions. A sphere-packing type upper bound naturally follows.

**Theorem 3**: Let $C \subseteq \mathbb{F}_2^n$ be a $(t,1)$-burst correcting code, then

$$|C| \leq \frac{2^{n-t+1}}{n-t+2}.$$

**Proof**: Given any two words $c_1, c_2 \in C$, $B_{t,1}(c_1) \cap B_{t,1}(c_2) = \emptyset$.

Now consider the union of all $(t,1)$-burst balls centered at the codewords in $C$. Obviously, their union is a subset of $\mathbb{F}_2^{n-t+1}$. In other words,

$$\left| \bigcup_{c \in C} B_{t,1}(c) \right| \leq 2^{n-t+1}.$$

Since any two $(t,1)$-burst balls centered at distinct codewords in $C$ are disjoint,

$$\left| \bigcup_{c \in C} B_{t,1}(c) \right| = \sum_{c \in C} |B_{t,1}(c)| = (n - t + 2) \cdot |C|$$

Consequently,

$$|C| \leq \frac{2^{n-t+1}}{n-t+2}.$$

Note that for $t = 1$ the upper bound above reduces to the Hamming bound for a single substitution. According to this sphere-packing type upper bound, the redundancy of a $(t,1)$-burst correcting code is lower bounded by

$$\log(n - t + 2) + t - 1 \approx \log n + t - 1.$$

**V. A General Construction of $(t,1)$-Burst Correcting Codes for Arbitrary $t$**

In this section, we provide a construction of $(t,1)$-burst correcting codes for arbitrary $t \geq 3$. We start with an overall explanation of the main idea. Then we proceed with the detailed construction and analysis, followed by some discussions.
A. Main Idea

From the representation of $A_{t-1}(x)$, it is not hard to find that a $(t, 1)$-burst will cause a $(2, 1)$-burst in the row where the starting coordinate of the $(t, 1)$-burst lies, and every other row suffers from a single deletion. Assuming the $(t, 1)$-burst starts at the $i$th coordinate, then the transformation of $x$ is as follows:

$$
\begin{bmatrix}
\cdots & x_{i-1} & x_{i+1} & \cdots \\
\cdots & x_{i-2} & x_{i+2} & \cdots \\
\cdots & x_{i-3} & x_{i+3} & \cdots \\
\cdots & \vdots & \vdots & \vdots \\
\cdots & x_{i+t-3} & x_{i+2t-4} & \cdots \\
\end{bmatrix} \rightarrow \begin{bmatrix}
\cdots & x_{i-1} & x_{i+t-2} & \cdots \\
\cdots & x_{i-2} & x_{i+t-1} & \cdots \\
\cdots & x_{i-3} & x_{i+t} & \cdots \\
\cdots & \vdots & \vdots & \vdots \\
\cdots & x_{i+t-3} & x_{i+2t-4} & \cdots \\
\end{bmatrix}.
$$

Note that the $(2, 1)$-burst correcting code as shown in Equation (2) can also correct a single deletion, due to the VT syndrome in its definition. Also note that the $t$ deletions occur within at most two columns in $A_{t-1}(x)$. Therefore, if we let the first row of $A_{t-1}(x)$ be a $(2, 1)$-burst correcting code, then decoding this row will provide us some additional knowledge about the location of the error in the other rows. To be more specific, the decoding of the first row (using a decoder for a $(2, 1)$-burst correcting code) falls into three cases.

1) The first row suffers from a $(2, 1)$-burst where the two deleted symbols are equal and different from the inserted symbol.
2) The first row suffers from a $(2, 1)$-burst which does not belong to Case 1.
3) The first row suffers from a single deletion.

For Case 1, we can uniquely determine the two erroneous coordinates in the first row, and thereby the erroneous coordinates in the other rows. Case 2 can be also seen as a single deletion in the first row. Therefore for both Case 2 and Case 3 we may locate the single deletion in the first row within a run, which might range from the $c_1$th to the $c_2$th column, for some $c_1 \leq c_2$. Then the errors in the other rows should be within columns $c_1 - 1$ to $c_2$ and we may apply a $(2, 1)$-burst-SVT code (to be defined later), which can correct a $(2, 1)$-burst with the additional knowledge about the location of the error within an interval of $P$ consecutive coordinates.

Here is a toy example to illustrate the idea above.

**Example 3:** Let $x = 101011001101110$, $t = 4$, and the erroneous sequence is $x' = 10101011110$. First write $x'$ in the form of an array,

$$A_3(x') = \begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}.
$$

Applying the decoder of a $(2, 1)$-burst correcting code to the first row, we get 10011. In addition we know that the first row suffers from a single deletion on either the second or the third coordinate. Therefore, the errors in the remaining two rows are within their first three coordinates. Applying the decoder of a $(2, 1)$-burst SVT code with $P = 3$ to them, we get the second and third rows as 01001, 11110 respectively. Consequently,

$$A_3(x) = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
\end{bmatrix},
$$

and $x = 101011001101110$ is correctly decoded.

If $P$ is large (e.g., linear in $n$), using $(2, 1)$-burst SVT codes does not reduce the total redundancy as compared to just using $(2, 1)$-burst correcting codes. Therefore, to further reduce the redundancy we want to make $P$ small, i.e., to make the length of every run in the first row limited. Hence, the $(d, k)$ run length limited (RLL) constraint is needed, which means that the length $\ell$ of any run satisfies $d \leq \ell \leq k$. Define $S_n(\ell)$ to be the set of binary sequences of length $n$ whose maximum run length is at most $\ell$. Pick $\ell = \log n + 3$. In [13] an algorithm is provided to efficiently encode/decode any binary sequence to a $(0, \log n + 3)$-RLL sequence with only 1 bit of redundancy. As a consequence, $|S_n(\log n + 3)| \geq 2^{n-1}$. Once we use a subcode of $S_n(\log n + 3)$ in the first row, it is guaranteed that for the other rows we have the additional knowledge about the location of the deleted symbol within an interval of $P = \log n + 4$ consecutive coordinates.

Now, we are fully prepared to present our main construction.

B. Construction

We follow the framework of writing $x$ as an array $A_{t-1}(x)$ of size $(t-1) \times \frac{n}{t-1}$. The first row $A_{t-1}(x)$ comes from the following code.

**Construction 1:** For arbitrary integers $n$ and $a \in \mathbb{Z}_{2n-1}$, $b \in \mathbb{Z}_4$, define the $C_{2,1}^{RLL}(n; a, b, \log n + 3)$ as

$$C_{2,1}^{RLL}(n; a, b, \log n + 3) = C_{2,1}(n; a, b) \cap S_n(\log n + 3).$$

**Theorem 4:** For arbitrary integers $n$, there exists a code $C_{2,1}^{RLL}(n; a, b, \log n + 3)$ with redundancy at most $\log n + 4$. 


Proof: Recall that $|S_n| = (log n + 3) \geq 2^{n-1}$. Moreover, for $a \in \mathbb{Z}_{2^{n-1}}, b \in \mathbb{Z}_4$, $\bigcup_{a,b} \mathcal{C}_{2,1}(n; a, b) \cap S_n$ is a disjoint partition of $S_n$. Thus according to the pigeonhole principle, there must exist choices for $a \in \mathbb{Z}_{2^{n-1}}$ and $b \in \mathbb{Z}_4$, such that

$$|\mathcal{C}_{2,1}(n; a, b, \log n + 3)| \geq \frac{2^{n-1}}{4(2n-1)}.$$ 

Therefore, the redundancy is at most

$$n - \log |\mathcal{C}_{2,1}(n; a, b, \log n + 3)| = \log(2n) + 1 < \log n + 4.$$

Due to the RLL constraint of the first row, the starting coordinate of the error on each remaining row will be limited to an interval of length $\log n + 4$. Now, we will provide a code which can correct $(2, 1)$-burst with this additional knowledge.

Construction 2: For arbitrary integers $n$ and $c \in \mathbb{Z}_{2^n-1}, d \in \mathbb{Z}_4$, define the $SVT_{2,1}^{burst}(n; c, d, P)$ as

$$SVT_{2,1}^{burst}(n; c, d, P) = \left\{ x : VT(x) \equiv c \pmod{2P-1}, \sum_{i=1}^{n} x_i \equiv d \pmod{4} \right\}.$$

Theorem 5: The $(2, 1)$-burst-SVT code $SVT_{2,1}^{burst}(n; c, d, P)$ can correct $(2, 1)$-burst with the additional knowledge of the starting location of the $(2, 1)$-burst within an interval of $P$ consecutive coordinates. Furthermore, there exist choices for $c$ and $d$ such that the redundancy of the code is at most $log P + 3$.

Proof: For any $a \in \mathbb{F}_2^n$ that suffers from a $(2, 1)$-burst, denote the received sequence as $u' \in \mathbb{F}_2^{n-1}$. Define $\Delta = \sum_{i=1}^{n} u_i - \sum_{i=1}^{n-1} u_i' \pmod{4}$. $\Delta \in \{0, 1, 2, 3\}$. The decoding starts with the observation of the value $\Delta$.

We have $\Delta = 3$ if and only if $0 \rightarrow 1$ happens. That is, the $(2, 1)$-burst error deletes two consecutive 0s and then inserts the symbol 1. Similarly, we have $\Delta = 2$ if and only if $1 \rightarrow 0$ happens. For the remaining cases, when $\Delta = 0$ or 1, the $(2, 1)$-burst error could be seen as just a single deletion. Since $SVT_{2,1}^{burst}(n; c, d, P)$ is a subcode of SVT code, it can correct a single deletion with the additional knowledge of the deleted coordinate within a length-$P$ interval.

We are only left with the decoding when $11 \rightarrow 0$ or $01 \rightarrow 0$ happens. We only prove the case for $11 \rightarrow 0$. The case $00 \rightarrow 1$ can be proved analogously.

Suppose there exist $x, y \in SVT_{2,1}^{burst}(n; c, d, P)$, such that $z \in B_{2,1}(x) \cap B_{2,1}(y)$. Suppose the $(2, 1)$-burst of the form $11 \rightarrow 0$ starts at $i$th coordinate of $x$ and $j$th coordinate of $y$, and without loss of generality $i < j$. We have

$$x = (x_1, \ldots, x_{i-1}, 1, 1, x_{i+2}, \ldots, x_j, 0, x_{j+2}, \ldots, x_n),$$

$$y = (y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_{j-1}, 1, 1, y_{j+2}, \ldots, y_n),$$

where $x_k = y_k$ when $1 \leq k \leq i - 1$ and $j + 2 \leq k \leq n$, $x_{k+1} = y_k$ when $i + 1 \leq k \leq j - 1$. Now we consider the difference of syndromes $VT(x) - VT(y)$, which is equal to

$$\sum_{i=1}^{n} ix_i - \sum_{i=1}^{n} iy_i = i + (i + 1) + wt(x_{i+2}, \ldots, x_j) - j - (j + 1) = 2(i - j) + wt(x_{i+2}, \ldots, x_j).$$

Since $0 \leq wt(x_{i+2}, \ldots, x_j) \leq j - i - 1$, we have

$$2(i - j) \leq VT(x) - VT(y) \leq i - j - 1.$$ 

Furthermore, $i$ and $j$ must be in an interval of length $P$, so $1 \leq j - i \leq P - 1$. Hence,

$$-2(P - 1) \leq VT(x) - VT(y) \leq -2.$$ 

Therefore, $VT(x) - VT(y) \neq 0$. which contradicts to the fact that $x, y \in SVT_{2,1}^{burst}(n; c, d, P)$. Thus $SVT_{2,1}^{burst}(n; c, d, P)$ can uniquely correct a $(2, 1)$-burst error with the additional knowledge of the starting location of the $(2, 1)$-burst within an interval of $P$ consecutive coordinates.

Moreover, since $\bigcup_{c,d} SVT_{2,1}^{burst}(n; c, d, P)$ is a partition of $\mathbb{F}_2^n$, according to the pigeonhole principle, there must exist $c$ and $d$ such that the code size is at least $\frac{2^n}{4P}$, thus the redundancy of the code is at most $\log(2P - 1) + 2 < \log P + 3$. 

Now, we are ready to present our construction of $(t, 1)$-burst correcting codes.

Construction 3: Let $a \in \mathbb{Z}_{2^n/t-1}, b \in \mathbb{Z}_4$, and $c_i \in \mathbb{Z}_{2^{t-1}}, d_i \in \mathbb{Z}_4$, where $2 \leq i \leq t - 1$, $P = \log \frac{n}{t-1} + 4$. The code $C_{t,1}$ is constructed as follows:

$$C_{t,1} \triangleq \left\{ x : A_{t-1}(x)_1 \in C_{2,1}^{RLL} \left( \frac{n}{t-1}; a, b, \log \frac{n}{t-1} + 3 \right) \right\},$$

$$A_{t-1}(x)_i \in SVT_{2,1}^{burst} \left( \frac{n}{t-1}; c_i, d_i, \log \frac{n}{t-1} + 4 \right),$$

for $2 \leq i \leq t - 1$. 


Theorem 6: The code $C_{t,1}$ is a $(t,1)$-burst correcting code, and there exist choices for $a, b, c, d$, such that the redundancy of the code is at most $\log n + (t-2) \log \log n + 4t - 4 - \log(t-1)$ when $n(t-1) \geq 16$, or at most $\log n + (t-2) \log \log n + 3t - 2 - \log(t-1)$ when $t \geq 17$.

Proof: Suppose $x \in C_{t,1}$ suffers from a $(t,1)$-burst and the obtained sequence is denoted as $y$. Write $y$ in the form of a $(t-1) \times \left(\frac{n}{t-1}\right)$ array $A_{t-1}(y)$. The $(t,1)$-burst will cause either a $(2,1)$-burst or a single deletion in the first row, which could be correctly decoded since the first row comes from a $(2,1)$-burst correcting code.

After decoding $A_{t-1}(y)$, due to the run-length limit of the first row, we will get some additional knowledge.

On one hand, if the first row suffers from a single deletion, then we may locate its deleted coordinate within an interval of length $\log \frac{n}{t-1} + 3$. The starting coordinate of error in the second to the $(t-1)$th row will be located within an interval of length $\log \frac{n}{t-1} + 4$. Then the rest rows can be uniquely decoded due to the property of the $(2,1)$-burst SVT code.

On the other hand, if the first row suffers from a $(2,1)$-burst in the form of $0 \rightarrow 1$ or $1 \rightarrow 0$, then we may even further locate the precise erroneous coordinates on all rows and then the second to the $(t-1)$th row will also be uniquely decoded.

To sum up, $C_{t,1}$ is indeed a $(t,1)$-burst correcting code.

For the size of the code, by the pigeonhole principle there must exist choices for $a, b, c, d$, $2 \leq i \leq t-1$, such that

$$C_{t,1} \geq \frac{2^n}{4 \cdot \frac{n}{t-1} \cdot \left(4 \cdot \left(2 \log \frac{n}{t-1} + 4\right) - 4\right)}$$

Hence, the redundancy is at most

$$2 + \log \frac{2n}{t-1} + (t-2)(2 + \log(2 \log \frac{n}{t-1} + 4)),$$

which is $\log n + (t-2) \log \log n + O(1)$. To be more specific, the redundancy is at most $\log n + (t-2) \log \log n + 4t - 4 - \log(t-1)$ when $n(t-1) \geq 16$, and at most $\log n + (t-2) \log \log n + 3t - 2 - \log(t-1)$ when $t \geq 17$.

C. Discussions

Note that, $(t,1)$-burst correcting codes can also correct $(t-1,0)$-burst errors (i.e., $(t,1)$-burst-deletion), then $(t,1)$-burst correcting codes are naturally also $(t-1)$-burst-deletion-correcting codes. Regarding the redundancy, the optimal redundancy of $(t,1)$-burst correcting codes should be lower bounded by the optimal redundancy of $(t-1)$-burst-deletion-correcting codes.

Up till now the best known construction of $(t-1)$-burst-deletion-correcting codes is the one from [13], where the redundancy is about $\log n + (t-2) \log \log n + O(1)$. Comparing with the redundancy of our $(t,1)$-burst correcting codes from Construction [3] there is only a difference of a constant term. Therefore, as a byproduct, our codes from Construction [3] also performs well against only $(t-1)$-burst-deletions.

Furthermore, note that there is only a $\log \log n$ gap between the redundancy of Construction [3] and the lower bound suggested by the sphere-packing bound. In the next section, for $t = 3$ we manage to close this gap.

VI. Optimal Codes Correcting $(3,1)$-Burst

Recall that the optimal $(2,1)$-burst correcting code with redundancy $\log n + 3$ is the one from [13], as shown in Equation [2]. This code adds an additional constraint on the basis of the VT code in order to deal with the type of errors that the VT code cannot correct. Motivated by this construction and the fact that a large proportion of $(3,1)$-burst errors could be seen as a 2-burst-deletion, we build our code based on Levenshtein’s code for 2-burst-deletions as shown in Equation [1], and add some other constraints in order to help deal with the other kinds of $(3,1)$-burst errors.

Construction 4: For $a \in \mathbb{Z}_4, b, c \in \mathbb{Z}_4$, and $d \in \mathbb{Z}_5$, the code $C_{3,1}$ is defined as follows:

$$C_{3,1}(n; a, b, c, d) = \left\{ x : Rsyn(0x) \equiv a \pmod{4n}, \sum_{i=1}^{n/2} x_{2i-1} \equiv b \pmod{4}, \sum_{i=1}^{n/2} x_{2i} \equiv c \pmod{4}, r(x) \equiv d \pmod{5} \right\}.$$
where $x$ by a 000 that the error is indeed a analyze the case when the error is of the form
\[ \{0, 0, 0, 1, (0, 1), (1, 0), (1, 1)\}. \] Thus, the disjointness of the possible values of $(\Delta_{odd}(x), \Delta_{even}(x))$ allows us to precisely determine the error pattern.

The first step of our decoding process is to observe $(\Delta_{odd}(x), \Delta_{even}(x))$ based on Lemma 3. Suppose that we have verified that the error is indeed a 2-burst-deletion. Then, as proved by Levenshtein [12], the constraint $Rsyn(0x) \equiv a \pmod{4n}$ in the code $C_{3,1}$ guarantees that we can successfully decode any 2-burst-deletion error. Therefore, from now on we only need to analyze the case when the error is of the form $\{000 \to 010, 011 \to 0, 101 \to 0\}$. We only need to prove that $C_{3,1}$ is an error-correcting code against the error type $000 \to 1$ or the error type $010 \to 1$. The case $111 \to 0$ and $101 \to 0$ can be proved analogously and are thus omitted.

**Lemma 4:** $C_{3,1}$ is an error-correcting code against 000 $\to 1$ errors.

**Proof:** Suppose we have two distinct codewords $x, y \in C_{3,1}(n; a, b, c, d)$ and $z \in F_2^n$ can be derived from both $x$ and $y$ by a 000 $\to 1$ error. Then $x$ and $y$ should be of the form
\[
\begin{align*}
x &= (x_1, \ldots, x_{i-1}, 0, 0, x_{i+2}, x_{i+3}, \ldots, x_n), \\
y &= (y_1, \ldots, y_{i-1}, 1, y_{i+2}, \ldots, 0, y_{j+1}, y_{j+2}, \ldots, y_n),
\end{align*}
\]
where $x_k = y_k$ when $1 \leq k \leq i - 1$ and $j + 3 \leq k \leq n$, $x_{i+2} = y_k$ when $i + 1 \leq k \leq j - 1$.

We turn to the last constraint $r(x)$ in the definition of $C_{3,1}$, which is the number of runs. Let $\Delta_r(x) = r(x) - r(z) \pmod{5}$ and $\Delta_r(y) = r(y) - r(z) \pmod{5}$. Since $x, y$ are both codewords from $C_{3,1}$, we must have $\Delta_r(x) = \Delta_r(y)$ and its value is of the following possibilities:

- If $i = 1$, i.e., the (3, 1)-burst starts at the very beginning, then depending on whether $x_4 = 0$ or $x_4 = 1$, the error pattern is either 0000 $\to 10$ or 0001 $\to 11$ and thus $\Delta_r(x)$ is 4 or 1.
- If $i = n - 2$, i.e., the (3, 1)-burst starts at the end, then depending on whether $x_{n-3} = 0$ or $x_{n-3} = 1$, the error pattern is either 0000 $\to 01$ or 1000 $\to 11$ and thus $\Delta_r(x)$ is 4 or 1.
- Otherwise, consider the four different cases for $x_{i-1}$ and $x_{i+3}$. The error pattern is 000000 $\to 010, 000001 \to 011, 100000 \to 110, 100011 \to 111$ and the corresponding value of $\Delta_r(x)$ is 3, 0, 2, and 0.

The rest of the proof falls into five cases. In each case we arrive at a contradiction and thus prove that the assumption of the two distinct codewords $x, y \in C_{3,1}(n; a, b, c, d)$ is not valid.

- **Case 1:** $\Delta_r(x) = \Delta_r(y) = 1$. Then according to the analysis above, we have

\[
\begin{align*}
x &= (0, 0, 0, 1, x_5, \ldots, x_{n-2}, 1, 1), \\
y &= (1, 1, y_3, \ldots, y_{n-4}, 1, 0, 0),
\end{align*}
\]
where $x_{k+2} = y_k$ when $3 \leq k \leq n - 4$ and the 000 $\to 1$ error starts at the first coordinate of $x$ and the $(n - 2)$th coordinate of $y$. Now we turn back to the constraint $Rsyn(0x)$ to derive a contradiction. The run sequence $R(0x)$ is of the form $x = \{(0, 0, 0, 1, -\text{run sequence corresponding to } x_5, \ldots, x_{n-2}) - (\lambda - 1, \lambda - 1) \}$ where $\lambda = r(0x) = r(x)$. The run sequence $R(y)$ is of the form $y = \{(0, 1, 1, -\text{run sequence corresponding to } y_3, \ldots, y_{n-4}) - (\lambda' - 1, \lambda' - 1, \lambda' - 1) \}$ where $\lambda' = r(0y) = r(y) + 1$. Note that for every $3 \leq k \leq n - 4$, the run index of $x_{k+2}$ in $0x$ equals the run index of $y_k$ in $Rsyn(0y)$. Also note that according to the definition of the code we must have $r(x) = r(y)$. Thus we have

\[
Rsyn(0x) - Rsyn(0y) = (2\lambda - 1) - (4\lambda' - 3) = -2r(x) - 2 \not\equiv 0 \pmod{4n},
\]
which is a contradiction to the condition that $Rsyn(0x) \equiv Rsyn(0y) \pmod{4n}$.

- **Case 2:** $\Delta_r(x) = \Delta_r(y) = 4$. Similarly as the previous case, we have

\[
\begin{align*}
x &= (0, 0, 0, 0, x_5, \ldots, x_{n-2}, 0, 0), \\
y &= (1, 0, y_3, \ldots, y_{n-4}, 0, 0, 0),
\end{align*}
\]
where $x_{k+2} = y_k$ when $3 \leq k \leq n - 4$ and the 000 $\to 1$ error starts at the first coordinate of $x$ and the $(n - 2)$th coordinate of $y$. The run sequence $R(0x)$ is of the form $x = \{(0, 0, 0, 0, 0) - (\text{run sequence corresponding to } x_5, \ldots, x_{n-2}) - (\lambda - 2, \lambda - 1) \}$ where $\lambda = r(0x) = r(x)$. The run sequence $R(y)$ is of the form $y = \{(0, 1, 2) - (\text{run sequence corresponding to } y_3, \ldots, y_{n-4}) - (\lambda' - 1, \lambda' - 1, \lambda' - 1, \lambda' - 1) \}$ where $\lambda' = r(0y) = r(y) + 1$. Moreover, for every $3 \leq k \leq n - 4$, the run index of $x_{k+2}$ in $0x$ is the run index of $y_k$ in $Rsyn(0y)$ minus two. Thus we have

\[
Rsyn(0x) - Rsyn(0y) = (2\lambda - 3) - 2(n - 6) - (4\lambda' - 1) = -2n - 2r(x) + 6 \pmod{4n}.
\]
From the representation of $x$, $2 \leq r(x) \leq n - 3$. Thus $-4n + 12 \leq -2n - 2r(x) + 6 \leq -2n + 2$ and thus $Rsyn(0x) - Rsyn(0y)$ is nonzero modulo 4n, a contradiction.
• Case 3: $\Delta_r(x) = \Delta_r(y) = 3$ and the error pattern must be $00000 \rightarrow 010$. We have

\[
x = (\ldots, r_i, 0, 0, 0, 0, x_{i+4}, \ldots, x_j, 0, 1, 0, \ldots),
\]

\[
y = (\ldots, r_i, 1, 0, t_{i+2}, y_{i+2}, \ldots, y_{j-2}, 0, 0, 0, 0, 0, r_{j+2}, \ldots),
\]

where the $r_i$'s are the run index of corresponding entries here. Moreover, for every $i + 2 \leq k \leq j - 2$, the run index of $x_{k+2}$ in $0x$ is the run index of $y_k$ in $Rsyn(0y)$ minus two. In this case,

\[
Rsyn(0x) - Rsyn(0y) = (5r_i + 3r_j + 3) - 2(j - i - 3) - (3r_i + 5r_j + 3) = 2(r_i - r_j) - 2(j - i) - 4 \quad (\text{mod } 4n).
\]

From the representation of $x$ we have $r_i \leq r_j \leq r_i + j - i - 2$. Hence, $-4(j - i) \leq Rsyn(0x) - Rsyn(0y) \leq -(2(j - i) - 4).$

Moreover, $2 \leq j - i \leq n - 5$, thus we have $-4(n - 5) \leq Rsyn(0x) - Rsyn(0y) \leq -8$, which is again a contradiction to $Rsyn(0x) - Rsyn(0y) \equiv 0 \pmod{4n}$.

• Case 4: $\Delta_r(x) = \Delta_r(y) = 2$ and the error pattern must be $10001 \rightarrow 111$. We have

\[
x = (\ldots, r_i, 1, 0, 0, 0, 1, x_{i+4}, \ldots, x_j, 1, 1, 1, \ldots),
\]

\[
y = (\ldots, r_i, 1, 1, y_{i+2}, \ldots, y_{j-2}, 1, 0, 0, 0, 1, r_{j+2}, \ldots),
\]

In this case, for every $i + 2 \leq k \leq j - 2$, the run index of $x_{k+2}$ in $0x$ is the run index of $y_k$ in $Rsyn(0y)$ plus two. Then we have

\[
Rsyn(0x) - Rsyn(0y) = (5r_i + 3r_j + 3) + 2(j - i - 3) - (3r_i + 5r_j - 5) = 2(r_i - r_j) + 2(j - i) + 4 \quad (\text{mod } 4n).
\]

From the representation of $x$ we have $r_i + 2 \leq r_j \leq r_i + j - i$. Hence, $4 \leq Rsyn(0x) - Rsyn(0y) \leq 2(j - i)$. Moreover, $2 \leq j - i \leq n - 5$, thus we have $4 \leq Rsyn(0x) - Rsyn(0y) \leq 2(n - 5)$, which is again a contradiction to $Rsyn(0x) - Rsyn(0y) \equiv 0 \pmod{4n}$.

• Case 5: $\Delta_r(x) = \Delta_r(y) = 0$. This case is further divided depending on the specific error pattern for $x$ and $y$. We only present one subcase as an example and the others can be proved analogously. Consider the subcase when both the error patterns in $x$ and $y$ are $00001 \rightarrow 011$. We have

\[
x = (\ldots, r_i, 0, 0, 0, 0, 1, x_{i+4}, \ldots, x_j, 0, 1, 1, \ldots),
\]

\[
y = (\ldots, 0, r_i, r_i, r_{i+1}, y_{i+2}, \ldots, y_{j-2}, 0, 0, 0, 0, 1, \ldots),
\]

Then we have

\[
Rsyn(0x) - Rsyn(0y) = (5r_i + 3r_j + 3) - (3r_i + 5r_j + 3) = 2(r_i - r_j) \quad (\text{mod } 4n).
\]

In this subcase, $r_i + 2 \leq r_j \leq r_i + j - i - 1$. Therefore, $-2(n - 6) \leq -2(j - i - 1) \leq Rsyn(0x) - Rsyn(0y) \leq -4$, again a contradiction.

To sum up, we have assumed that we have two distinct codewords $x, y \in C_{3,1}(a, b, c, d)$ and $z \in \mathbb{F}_2^n$ can be derived from both $x$ and $y$ by a $000 \rightarrow 1$ error. However, in all cases we can deduce that $Rsyn(0z) - Rsyn(0y)$ is nonzero modulo $4n$, which contradicts to the definition of the code. Thus, we have proven that $C_{3,1}$ is an error-correcting code against $000 \rightarrow 1$ errors.

Lemma 5: $C_{3,1}$ is an error-correcting code against $010 \rightarrow 1$ errors.

Proof: The proof follows the same way as the previous lemma. Assume that we have two distinct codewords $x, y \in C_{3,1}(a, b, c, d)$ and $z \in \mathbb{F}_2^n$ can be derived from both $x$ and $y$ by a $010 \rightarrow 1$ error. $x$ and $y$ should be of the form

\[
x = (x_1, \ldots, x_{i-1}, 0, 1, 1, 0, x_{i+3}, \ldots, x_{j+1}, 1, x_{j+3}, \ldots, x_n),
\]

\[
y = (y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_{j-1}, 0, 1, 0, y_{j+3}, \ldots, y_n),
\]

where $x_k = y_k$ when $1 \leq k \leq i - 1$ and $j + 3 \leq k \leq n$, $x_{k+2} = y_k$ when $i + 1 \leq k \leq j - 1$. First we need to observe the two values $\Delta_r(x), \Delta_r(y)$.

• If $i = 1$, i.e., the (3, 1)-burst starts at the very beginning, then depending on whether $x_4 = 0$ or $x_4 = 1$, the error pattern is either $0100 \rightarrow 10$ or $0101 \rightarrow 11$ and thus $\Delta_r(x)$ is 1 or 3.

• If $i = n - 2$, i.e., the (3, 1)-burst starts at the end, then depending on whether $x_{n-3} = 0$ or $x_{n-3} = 1$, the error pattern is either $0100 \rightarrow 01$ or $1010 \rightarrow 11$ and thus $\Delta_r(x)$ is 1 or 3.

• Otherwise, consider the four different cases for $x_{i+1}$ and $x_{i+2}$. The error pattern is $00100 \rightarrow 010, 00101 \rightarrow 011, 10100 \rightarrow 110, 10101 \rightarrow 111$ and the corresponding value of $\Delta_r(x)$ is 0, 2, 2, and 4.

Based on the observations of $\Delta_r(x), \Delta_r(y)$ we then break into several cases and in each case we will derive a contradiction by analyzing $Rsyn(0x) - Rsyn(0y)$. Since the whole framework is similar as the previous lemma, we omit some details and only present the calculations for $Rsyn(0x) - Rsyn(0y)$. 


• Case 1: \( \Delta_r(x) = \Delta_r(y) = 1 \).
\[ x = (0, 1, 0, 0, x_5, \ldots, x_{n-2}, 0, 1), \quad y = (1, 0, y_3, \ldots, y_{n-4}, 0, 0, 1, 0). \]

Then \( R_{syn}(0x) - R_{syn}(0y) = -2r(x) + 4 \neq 0 \), since \( r(x) \geq 4 \).

• Case 2: \( \Delta_r(x) = \Delta_r(y) = 3 \).
\[ x = (0, 1, 0, 1, x_5, \ldots, x_{n-2}, 1, 1), \quad y = (1, 1, y_3, \ldots, y_{n-4}, 1, 0, 1). \]

This case is a little bit special and deserves to be analyzed in detail. We can compute that \( R_{syn}(0x) - R_{syn}(0y) = 2n - 2r(x) - 4 \). Suppose it is zero modulo 4\( n \), then \( n = r(x) + 2 \). Since the first symbol of \( x \) is 0 and the last symbol of \( x \) is 1, then the number of runs \( r(x) \) must be even and thereby \( n \) must be even. As \( \sum_{k=1}^{n/2} x_{2k-1} = 1 + \sum_{k=3}^{n/2-1} x_{2k-1}, \sum_{k=1}^{n/2} y_{2k-1} = 3 + \sum_{k=2}^{n/2-1} y_{2k-1} \), and \( \sum_{k=2}^{n/2-1} x_{2k-1} = \sum_{k=2}^{n/2-2} y_{2k-1} \), we get \( \sum_{k=1}^{n/2} x_{2k-1} - \sum_{k=1}^{n/2} y_{2k-1} = 2 \) (mod 4) which is a contradiction to \( \sum_{k=1}^{n/2} x_{2k-1} - \sum_{k=1}^{n/2} y_{2k-1} \equiv 0 \) (mod 4). Therefore, \( R_{syn}(0x) - R_{syn}(0y) = 2n - 2r(x) - 4 \neq 0 \) (mod 4).

• Case 3: \( \Delta_r(x) = \Delta_r(y) = 0 \) and the error pattern must be 00100 \( \to 010 \). Hence,
\[ x = (\ldots, r_i, 0, r_j, r_i+1, r_j+2, r_i, r_j, \ldots), \quad y = (\ldots, 0, 1, 0, r_i+2, r_i+1, r_i, r_j, 0, 0, 1, 0, 0, \ldots), \]

Then \( R_{syn}(0x) - R_{syn}(0y) = 2(r_i - r_j) \neq 0 \) (mod 4).

• Case 4: \( \Delta_r(x) = \Delta_r(y) = 4 \) and the error pattern must be 10101 \( \to 111 \). Hence,
\[ x = (\ldots, 1, 0, 1, 0, x_{i+4}, \ldots, x_j, 0, 1, 0, \ldots), \quad y = (\ldots, 1, 0, 1, 0, 1, 1, 1, 1, \ldots), \]

\[ R_{syn}(0x) - R_{syn}(0y) = 2(r_i - r_j) + 4(j - i) + 8. \] Since \( r_i + 4 \leq r_j \leq r_i + j - i + 2 \), we have
\[ 2(j - i) + 4 \leq R_{syn}(0x) - R_{syn}(0y) \leq 4(j - i), \]
so \( R_{syn}(0x) - R_{syn}(0y) \neq 0 \) (mod 4).

• Case 5: \( \Delta_r(x) = \Delta_r(y) = 2 \). This case is further divided depending on the specific error pattern for \( x \) and \( y \). We only present one subcase as an example and the others can be proved analogically. Consider the subcase when the error pattern in \( x \) is 00101 \( \to 011 \) and the error pattern in \( y \) is 10100 \( \to 110 \). We have
\[ x = (\ldots, r_i, 0, 1, 0, 1, x_{i+4}, \ldots, x_j, 1, 1, 1, \ldots), \quad y = (\ldots, r_i, 1, 1, r_j, r_j, \ldots), \]

\[ R_{syn}(0x) - R_{syn}(0y) = 2(r_i - r_j) + 2(j - i). \] Suppose it is zero, then \( r_j - r_j = j - i \). Since the symbols with run index \( r_i \) are 0s and symbols with run index \( r_j \) are 1s, \( r_j - r_j \) must be odd and thus \( j - i \) is odd. The number of symbols from \( x_{i+4} \) to \( x_j \) is \( j - i - 3 \) and thus is even. WLOG let \( i \) be odd, then
\[ \sum_{k=1}^{n/2} x_{2k-1} - \sum_{k=1}^{n/2} y_{2k-1} = 1 - 3 \equiv 2 \] (mod 4),
which is a contradiction to \( \sum_{k=1}^{n/2} x_k - \sum_{k=1}^{n/2} y_k \equiv 0 \) (mod 4). As a consequence, \( R_{syn}(0x) - R_{syn}(0y) \neq 0 \) (mod 4).

To sum up, we have assumed that we have two distinct codewords \( x, y \in C_{3,1}(n; a, b, c, d) \) and \( z \in \mathbb{F}_2^n \) can be derived from both \( x \) and \( y \) by a 010 \( \rightarrow \) 1 error. However, in all cases we can deduce that \( R_{syn}(0x) - R_{syn}(0y) \) is nonzero modulo 4\( n \), which contradicts to the definition of the code. Thus, we have proven that \( C_{3,1} \) is an error-correcting code against 010 \( \rightarrow \) 1 errors.

We close this section by summarizing our construction of the code \( C_{3,1}(n; a, b, c, d) \):

- The decoding process starts with the observations of \( \Delta_{odd}(u) \) and \( \Delta_{even}(u) \).
- If we determine that the error can be seen as a 2-burst-deletion, then according to Levenshtein \cite{12} our code is capable of correcting a 2-burst-deletion error.
- Otherwise, we can determine the error pattern, which is one out of \( \{000 \rightarrow 1, 010 \rightarrow 1, 111 \rightarrow 0, 101 \rightarrow 0\} \).
- Whatever the error pattern is, our code can correct this type of error (two models are proved via Lemmas \cite{4} and \cite{5} and the other two follow a similar idea).

Finally, by the pigeonhole principle, we may find suitable parameters \( a \in \mathbb{Z}_{4n}, b, c \in \mathbb{Z}_4, \) and \( d \in \mathbb{Z}_5 \) and find a code with size at least \( \frac{2^n}{4n^3} \) and thus its redundancy is at most \( \log(320n) < \log n + 9 \). Note that the lower bound of the redundancy
of \((3, 1)\)-burst correcting codes is \(\log n + 2\), so our construction is optimal up to a constant. In sum, in this section we have proved the following.

**Theorem 7:** There exist choices of \(a \in \mathbb{Z}_{4n}, b, c \in \mathbb{Z}_4,\) and \(d \in \mathbb{Z}_5\), such that the code \(C_{3,1}(n; a, b, c, d)\) is a \((3, 1)\)-burst correcting code with redundancy at most \(\log n + 9\).

**VII. Conclusion**

In this paper we study \((t, 1)\)-burst correcting codes. First we prove the equivalence between \((t, 1)\)-burst correcting codes and \((1, t)\)-burst correcting codes. Then we present a sphere-packing type upper bound of \((t, 1)\)-burst correcting codes, leading to a lower bound of its redundancy. We present a construction of \((t, 1)\)-burst correcting codes for arbitrary \(t\), with redundancy \(\log n + (t - 2) \log \log n + O(1)\). Comparing our general construction and the lower bound of redundancy, there is only a \(\log \log n\) gap. We manage to close this gap for \(t = 3\) by giving a construction of \((3, 1)\)-burst correcting codes with redundancy at most \(\log n + 9\). For \(t \geq 4\), closing the gap is left for future research.

**References**


