CONCENTRATING SOLUTIONS FOR AN ANISOTROPIC PLANAR ELLIPTIC
NEUMANN PROBLEM WITH HARDY-HÉNON WEIGHT AND LARGE EXPONENT

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Abstract. Let Ω be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary, we study the following anisotropic elliptic Neumann problem with Hardy-Hénon weight

\[
\begin{aligned}
-\nabla (a(x) \nabla u) + a(x) u &= a(x) |x - q|^{2\alpha} u^p, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( q \in \overline{\Omega} \), \( \alpha \in (-1, +\infty) \setminus \mathbb{N}, \) \( p > 1 \) is a large exponent and \( a(x) \) is a positive smooth function. We investigate the effect of the interaction between anisotropic coefficient \( a(x) \) and singular source \( q \) on the existence of concentrating solutions. We show that if \( q \in \Omega \) is a strict local maximum point of \( a(x) \), there exists a family of positive solutions with arbitrarily many interior spikes accumulating to \( q \); while if \( q \in \partial \Omega \) is a strict local maximum point of \( a(x) \) and satisfies \( \langle \nabla a(q), \nu(q) \rangle = 0 \), such a problem has a family of positive solutions with arbitrarily many mixed interior and boundary spikes accumulating to \( q \). In particular, we find that concentration at singular source \( q \) is always possible whether \( q \in \Omega \) is an isolated local maximum point or not.

1. Introduction

This paper deals with the existence and profile of solutions for the following anisotropic elliptic Neumann problem

\[
\begin{aligned}
-\nabla (a(x) \nabla u) + a(x) u &= a(x) |x - q|^{2\alpha} u^p, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) with smooth boundary, \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( q \in \overline{\Omega} \), \( \alpha \in (-1, +\infty) \setminus \mathbb{N}, \) \( p > 1 \) is a large exponent and \( a(x) \) is a positive smooth function over \( \overline{\Omega} \). The term \( | \cdot |^{2\alpha} \) in equation (1.1) is called the Hardy weight if \(-1 < \alpha < 0\), whereas the Hénon weight if \( \alpha > 0 \) (see [19, 20]). We are interested in solutions of problem (1.1) which exhibit the concentration phenomenon as the exponent \( p \) tends to infinity.

This work is strongly motivated by some extensive research involving the case \( \alpha = 0 \) in equation (1.1):

\[
\begin{aligned}
-\nabla (a(x) \nabla u) + a(x) u &= a(x) u^p, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

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where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \) and \( p > 1 \). Equation (1.2) has a strong biological meaning because it arises from the study of steady states for the logarithmic Keller-Segel system in chemotaxis (see [21]):

\[
\begin{align*}
C_1 \Delta \psi - \chi \nabla \cdot (\psi \nabla \log \omega) &= 0 \quad \text{in} \quad \mathcal{D}, \\
C_2 \Delta \omega - a \omega + b \psi &= 0 \quad \text{in} \quad \mathcal{D}, \\
\frac{\partial \omega}{\partial \nu} = \frac{\partial \psi}{\partial \nu} &= 0 \quad \text{on} \quad \partial \mathcal{D}, \\
\frac{1}{|\mathcal{D}|} \int_{\mathcal{D}} \psi(x) dx &= \bar{\psi} > 0 \quad \text{(prescribed)},
\end{align*}
\]

(1.3)

where \( \mathcal{D} \) is a smooth bounded domain in \( \mathbb{R}^N (N \geq 2) \) and the constants \( C_1, C_2, a, b \) and \( \chi \) are positive. Testing the first equation in (1.3) against \( \nabla (\log \psi - p \log \omega) \) with \( p = \chi / C_1 \), we find

\[
\int_{\mathcal{D}} \psi |\nabla (\log \psi - p \log \omega)|^2 = 0,
\]

i.e. \( \psi = \lambda \omega^p \) for some constant \( \lambda > 0 \). Furthermore, setting \( \varepsilon^2 = C_2 / a, \gamma = (b \lambda / a)^{1/p} \) and \( \nu = \gamma \omega \), we have that \( \nu \) satisfies the Lin-Ni-Takagi problem in [22, 29, 30], namely the singularly perturbed elliptic Neumann equation

\[
\begin{align*}
-\varepsilon^2 \Delta \nu + \nu &= \nu^p, \quad \nu > 0 \quad \text{in} \quad \mathcal{D}, \\
\frac{\partial \nu}{\partial \nu} &= 0 \quad \text{on} \quad \partial \mathcal{D}.
\end{align*}
\]

(1.4)

This equation has attracted considerable attention in the past three years because its solutions exhibit a variety of concentration phenomena not only at one or more points but also on higher dimensional subsets of \( \mathcal{D} \) as either \( \varepsilon \) tends to zero or \( p \) approaches the \((h + 1)\)-th critical exponent \( p^{*}_{h+1} \), where \( p^{*}_{N-1} = +\infty \) and \( p^{*}_{h+1} = (N - h + 2)/(N - h - 2) \) for any \( 0 \leq h \leq N - 3 \). The reader can refer to [8, 17, 18] for the subcritical case \( p < p^{*}_{1} \), to [1, 32, 35, 36] for the critical case \( p = p^{*}_{1} \), to [7, 11, 28, 33, 34] for the almost first critical case \( p = p^{*}_{1} + d \) with \( 0 < d \to 0 \), to [23, 24, 25, 26] for the \((h + 1)\)-th subcritical case \( p < p^{*}_{h+1} \) with \( 1 \leq h \leq N - 2 \), to [10] for the \((h + 1)\)-th critical case \( p = p^{*}_{h+1} \) with \( 1 \leq h \leq N - 7 \), and to [12] for the almost \((h + 1)\)-th critical case \( p = p^{*}_{h+1} + d \) with \( 1 \leq h \leq N - 7 \) and \( 0 < d \to 0 \). In particular, problem (1.4) admits a solution with arbitrarily many mixed interior and boundary spikes, which has been shown in [17] for \( p < p^{*}_{1} \) fixed but \( \varepsilon > 0 \) small enough, and in [28] for \( \varepsilon = 1 \) but \( p \to p^{*}_{1} = +\infty \) with \( N = 2 \).

It is very interesting to point out that as a slight but natural generalization of (1.4)|\( \varepsilon=1 \), equation (1.2) can also be viewed as a special case of (1.4)|\( \varepsilon=1 \) when the domain \( \mathcal{D} \) has some rotational symmetries. Indeed, take \( n \geq 2 \) and \( n \geq m \geq 1 \) as fixed integers. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \) such that

\[
\begin{align*}
\bar{\Omega} \subset \{(x_1, \ldots, x_m, x') \in \mathbb{R}^n \times \mathbb{R}^{n-m} | x_i > 0, \quad i = 1, \ldots, m\}.
\end{align*}
\]

Fix \( k_1, \ldots, k_m \in \mathbb{N} \setminus \{0\} \) with \( h := k_1 + \cdots + k_m \) and set

\[
\mathcal{D} := \{(y_1, \ldots, y_m, x') \in \mathbb{R}^{k_1+1} \times \cdots \times \mathbb{R}^{k_m+1} \times \mathbb{R}^{n-m} | (|y_1|, \ldots, |y_m|, x') \in \bar{\Omega}\}.
\]

Then \( \mathcal{D} \) is a smooth bounded domain in \( \mathbb{R}^N \) with \( N := h + n \), which is invariant under the action of the group

\[
\begin{align*}
\mathcal{Y} := \mathcal{O}(k_1+1) \times \cdots \times \mathcal{O}(k_m+1) \quad \text{on} \quad \mathbb{R}^N \quad \text{given by}
\end{align*}
\]

\[
\begin{align*}
(g_1, \ldots, g_m)(y_1, \ldots, y_m, x') := (g_1 y_1, \ldots, g_m y_m, x'),
\end{align*}
\]

for each \( g_i \in \mathcal{O}(k_i+1), \ y_i \in \mathbb{R}^{k_i+1} \) and \( x' \in \mathbb{R}^{n-m} \). Note that \( \mathcal{O}(k_i+1) \) is the group of linear isometries of \( \mathbb{R}^{k_i+1} \) and \( \mathbb{S}^{k_i} \) is the unit sphere in \( \mathbb{R}^{k_i+1} \). For any \( p > 1 \) we shall seek \( \mathcal{Y} \)-invariant solutions of problem (1.4)|\( \varepsilon=1 \), i.e. solutions \( \nu \) of the form

\[
\nu(y_1, \ldots, y_m, x') = u(|y_1|, \ldots, |y_m|, x').
\]

(1.5)
A simple calculation implies that $v$ solves problem $(1.4)|_{\varepsilon=1}$ if and only if $u$ satisfies
\[
\begin{cases}
-\Delta u - \sum_{i=1}^{m} k_i \frac{\partial u}{x_i \partial x_i} + u = u^p, & u > 0 \text{ in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{ on } \partial \Omega.
\end{cases}
\] (1.6)

Thus if we take anisotropic coefficient
\[ a(x) = a(x_1, \ldots, x_m, x') := x_1^{k_1} \cdots x_m^{k_m}, \] (1.7)
then problem (1.6) can be rewritten as equation (1.2). Hence by considering rotational symmetry of $D$, a fruitful approach for seeking solutions of problem $(1.4)|_{\varepsilon=1}$ with concentration along some $h$-dimensional minimal submanifolds of $\mathcal{D}$ diffeomorphic to $S^{k_1} \times \cdots \times S^{k_m}$ is to reduce it to produce point-wise spiky or bubbling solutions of the anisotropic equation (1.2) in the domain $\Omega$ of lower dimension. This approach, together with some finite dimensional reduction arguments, has recently been taken to construct solutions of $(1.4)|_{\varepsilon=1}$ concentrating along an $h$-dimensional minimal submanifold of $\partial \mathcal{D}$, which can be found in [27] for the almost $(h+1)$-th critical case $p = p_{h+1}^* + d$ with $1 \leq h \leq N - 3$ and $0 < d \to 0$, and in [39] for the slightly $(N - 1)$-th subcritical case $p \to p_{N-1}^* = +\infty$.

This paper is devoted to studying the existence and concentration behavior of spiky solutions to the anisotropic planar equation (1.1) with large exponent $p$ and Hardy-Hénon weight $|x-q|^{2a}$ involving $a \in (-1, +\infty) \setminus \mathbb{N}$ only. We try to use a finite dimensional reduction to investigate the effect of the interaction between anisotropic coefficient $a(x)$ and singular source $q$ on the existence of concentrating solutions to problem (1.1) when $p$ goes to $+\infty$. As a result, we prove that if $q \in \Omega$ is a strict local maximum point of $a(x)$, there exists a family of positive solutions with arbitrarily many interior spikes accumulating to $q$; while if $q \in \partial \Omega$ is a strict local maximum point of $a(x)$ and satisfies $\partial_q a(q) := (\nabla a(q), \nu(q)) = 0$, such a problem has a family of positive solutions with arbitrarily many mixed interior and boundary spikes accumulating to $q$. To state our results more precisely, we first introduce some notations. Let
\[ \Delta_\alpha u = \frac{1}{a(x)} \nabla(a(x) \nabla u) = \Delta u + \nabla \log a(x) \nabla u \]
and $G(x, y)$ be the Green’s function satisfying
\[
\begin{cases}
-\Delta_\alpha G(x, y) + G(x, y) = \delta_y(x), & x \in \Omega, \\
\frac{\partial G}{\partial \nu_x}(x, y) = 0, & x \in \partial \Omega,
\end{cases}
\] (1.8)
for each $y \in \overline{\Omega}$, then its regular part is defined depending on whether $y$ lies in the domain or on its boundary as
\[
H(x, y) = \begin{cases}
G(x, y) + \frac{1}{2\pi} \log |x - y|, & y \in \Omega, \\
G(x, y) + \frac{1}{\pi} \log |x - y|, & y \in \partial \Omega.
\end{cases}
\] (1.9)

**Theorem 1.1.** Let $\alpha \in (-1, +\infty) \setminus \mathbb{N}$ and assume that $q \in \Omega$ is a strict local maximum point of $a(x)$. Then for any integer $m \geq 1$, there exists $p_m > 0$ such that for any $p > p_m$, problem (1.1) has a family of positive solutions $u_p$ with $m+1$ different interior spikes which accumulate to $q$ as $p \to +\infty$. More precisely,
\[
u_p(x) = \sum_{i=1}^{m} \frac{1}{\gamma \mu_i^{2/(p-1)} |\xi_i - q|^{2a/(p-1)}} \left[ \log \left( \frac{1}{(\varepsilon^2 \mu_i^2 + |x - \xi_i|^2)^2} + 8\pi H(x, \xi_i) \right) \right]^{1/2} + \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ \log \left( \frac{1}{(\varepsilon^2 \mu_0^2 + |x - q|^{2(1+\alpha)})^2} + 8\pi(1 + \alpha) H(x, q) \right) \right] + o(1),
\]
where $o(1) \to 0$, as $p \to +\infty$, on each compact subset of $\overline{\Omega} \setminus \{q, \xi^p_1, \ldots, \xi^p_m\}$, the parameters $\gamma$, $\varepsilon$, $\mu_0$ and $\mu_i$, $i = 1, \ldots, m$ satisfy
\[
\gamma = p^{p/(p-1)}e^{2/(p-1)}, \quad \varepsilon = e^{-p/4}, \quad \frac{1}{C} < \mu_0 < Cp^C, \quad \frac{1}{C} < \mu_i < Cp^C,
\]
for some $C > 0$, and $(\xi^p_1, \ldots, \xi^p_m) \in (\Omega \setminus \{q\})^m$ satisfies
\[
\xi^p_i \to q \quad \text{for all } i, \quad \text{and} \quad |\xi^p_i - \xi^p_k| > 1/p^{2(m+1+\alpha)^2} \quad \forall \ i \neq k.
\]
In particular, for any $d > 0$, as $p \to +\infty$,
\[
apa(x)|x - q|^{2\alpha}u^p_{p+1} \to 8\pi e(m + 1 + \alpha)a(q)\delta_q \quad \text{weakly in the sense of measure in } \overline{\Omega},
\]
but
\[
\sup_{x \in B_{1/p^{(m+1+\alpha)^2}}(q)} u_p(x) \to \sqrt{c} \quad \text{and} \quad \sup_{x \in B_{1/p^{(m+1+\alpha)^2}}(\xi^p_i)} u_p(x) \to \sqrt{c}, \quad i = 1, \ldots, m.
\]

**Theorem 1.2.** Let $\alpha \in (-1, +\infty) \setminus \mathbb{N}$ and assume that $q \in \partial\Omega$ is a strict local maximum point of $a(x)$ and satisfies $\partial_\nu a(q) := (\nabla a(q), \nu(q)) = 0$. Then for any integers $m \geq 1$ and $0 \leq l \leq m$, there exists $p_m > 0$ such that for any $p > p_m$, problem (1.1) has a family of positive solutions $u_p$ with $m - l + 1$ different boundary spikes and $l$ different interior spikes which accumulate to $q$ as $p \to +\infty$. More precisely,
\[
\sum_{i=1}^m \frac{1}{\gamma \mu_i^{2/(p-1)}} \left[ \log \frac{\varepsilon^2 p^2 + |x - \xi^p_i|^2}{\xi^p_i - q|^{2\alpha/(p-1)}} + c_i H(x, \xi^p_i) \right] + \sup_{x \in B_{1/p^{(m+1+\alpha)^2}}(q)} u_p(x) \to \sqrt{c},
\]
where $o(1) \to 0$, as $p \to +\infty$, on each compact subset of $\overline{\Omega} \setminus \{q, \xi^p_1, \ldots, \xi^p_m\}$, the parameters $\gamma$, $\varepsilon$, $\mu_0$ and $\mu_i$, $i = 1, \ldots, m$ satisfy
\[
\gamma = p^{p/(p-1)}e^{2/(p-1)}, \quad \varepsilon = e^{-p/4}, \quad \frac{1}{C} < \mu_0 < Cp^C, \quad \frac{1}{C} < \mu_i < Cp^C,
\]
for some $C > 0$, $(\xi^p_1, \ldots, \xi^p_m) \in \Omega^l \times (\partial\Omega)^{m-l}$ satisfies
\[
\xi^p_i \to q \quad \forall \ i, \quad |\xi^p_i - \xi^p_k| > 1/p^{2(m+1+\alpha)^2} \quad \forall \ i \neq k, \quad \text{and} \quad \text{dist}(\xi^p_i, \partial\Omega) > 1/p^{2(m+1+\alpha)^2} \quad \forall \ i = 1, \ldots, l,
\]
and $c_i = 8\pi$ for $i = 1, \ldots, l$, while $c_i = 4\pi$ for $i = l + 1, \ldots, m$. In particular, for any $d > 0$, as $p \to +\infty$,
\[
apa(x)|x - q|^{2\alpha}u^p_{p+1} \to 4\pi e(m + 1 + \alpha)a(q)\delta_q \quad \text{weakly in the sense of measure in } \overline{\Omega},
\]
but
\[
\sup_{x \in \overline{\Omega} \setminus B_{1/p^{(m+1+\alpha)^2}}(q)} u_p(x) \to \sqrt{c} \quad \text{and} \quad \sup_{x \in \overline{\Omega} \setminus B_{1/p^{(m+1+\alpha)^2}}(\xi^p_i)} u_p(x) \to \sqrt{c}, \quad i = 1, \ldots, m.
\]

In fact, the assumption in Theorem 1.2 can be split into the following two cases:

(C1) $q \in \partial\Omega$ is a strict local maximum point of $a(x)$ restricted on $\partial\Omega$;

(C2) $q \in \partial\Omega$ is a strict local maximum point of $a(x)$ restricted in $\Omega$ and satisfies $\partial_\nu a(q) := (\nabla a(q), \nu(q)) = 0$.

Arguing exactly along the sketch of the proof of Theorem 1.2, we can easily find that if (C1) holds, then problem (1.1) has solutions with arbitrarily many boundary spikes which accumulate to $q$ along $\partial\Omega$; while if (C2) holds, then problem (1.1) has solutions with arbitrarily many interior spikes which accumulate to $q$ along the inner normal direction of $\partial\Omega$.

For the case $m = 0$, we have the following two results which correspond to Theorems 1.1 and 1.2, respectively.
Theorem 1.3. Let $\alpha \in (-1, +\infty) \setminus \mathbb{N}$ and $q \in \Omega$. Then there exists $p_0 > 0$ such that for any $p > p_0$, problem (1.1) has a family of positive solutions $u_p$ such that as $p$ tends to $+\infty$,

$$u_p(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ \log \frac{1}{\varepsilon^2 \mu_0^2 + |x - q|^{2(1+\alpha)}} + 8\pi (1 + \alpha)H(x, q) \right] + o(1),$$

uniformly on each compact subset of $\overline{\Omega} \setminus \{q\}$, where the parameters $\gamma$, $\varepsilon$ and $\mu_0$ satisfy

$$\gamma = p^{p/(p-1)} \varepsilon^{2/(p-1)}, \quad \varepsilon = e^{-p/4}, \quad 1/C < \mu_0 < C,$$

for some $C > 0$. In particular, for any $d > 0$, as $p \to +\infty$,

$$pa(x)|x - q|^{2\alpha} u_p^{p+1} \to 8\pi e(1 + \alpha) a(q)\delta_q \quad \text{weakly in the sense of measure in } \overline{\Omega},$$

$$u_p \to 0 \quad \text{uniformly in } \overline{\Omega} \setminus B_d(q),$$

but

$$\sup_{x \in B_d(q)} u_p(x) \to \sqrt{\varepsilon}.$$

Theorem 1.4. Let $\alpha \in (-1, +\infty) \setminus \mathbb{N}$ and $q \in \partial \Omega$. Then there exists $p_0 > 0$ such that for any $p > p_0$, problem (1.1) has a family of positive solutions $u_p$ such that as $p$ tends to $+\infty$,

$$u_p(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ \log \frac{1}{\varepsilon^2 \mu_0^2 + |x - q|^{2(1+\alpha)}} + 4\pi (1 + \alpha)H(x, q) \right] + o(1),$$

uniformly on each compact subset of $\overline{\Omega} \setminus \{q\}$, where the parameters $\gamma$, $\varepsilon$ and $\mu_0$ satisfy

$$\gamma = p^{p/(p-1)} \varepsilon^{2/(p-1)}, \quad \varepsilon = e^{-p/4}, \quad 1/C < \mu_0 < C,$$

for some $C > 0$. In particular, for any $d > 0$, as $p \to +\infty$,

$$pa(x)|x - q|^{2\alpha} u_p^{p+1} \to 4\pi e(1 + \alpha) a(q)\delta_q \quad \text{weakly in the sense of measure in } \overline{\Omega},$$

$$u_p \to 0 \quad \text{uniformly in } \overline{\Omega} \setminus B_d(q),$$

but

$$\sup_{x \in \overline{\Omega} \setminus B_d(q)} u_p(x) \to \sqrt{\varepsilon}.$$

Let us comment that for the case $m = 0$, by arguing simply along some initial procedures of the construction of solutions of problem (1.1) we can prove the corresponding results in Theorems 1.3 and 1.4. This shows that problem (1.1) always admits a family of positive solutions concentrating only at singular source $q$ whether $q$ is an isolated local maximum point of $a(x)$ or not.

Finally, let us remark about the analogy and difference existing between our results and those known for the planar Neumann problem with Hardy-Hénon weight and large exponent

$$\begin{cases}
-\Delta u + u = |x - q|^{2\alpha} u^p, & u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.10)$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^2$, $\alpha \in (-1, +\infty) \setminus \mathbb{N}$, $q \in \overline{\Omega}$, $p$ is a large exponent and $\nu$ denotes the outer unit normal vector to $\partial \Omega$. Just like that in equation (1.1), the presence of Hardy-Hénon weight can also produce significant influence on the existence of a solution of problem (1.10) with spiky profile at each singular source $q \in \overline{\Omega}$, which has been proven in [38] that problem (1.10) always admits a family of positive solutions $u_p$ with arbitrarily many mixed interior and boundary spikes involving any $q \in \overline{\Omega}$ when $p$ tends to $+\infty$. However, due to the occurrence of Hardy-Hénon weight, it is necessary to point out that although the anisotropic planar equation (1.1) is seemingly similar to problem (1.10), equation (1.1) can not be viewed as a special version of (1.10) in higher dimension even if the domain has some rotational symmetries. This seems to imply that unlike
those for solutions of problem (1.10) in [38] whose multiple spikes are completely determined by the geometry of the domain, the location of multiple spikes in solutions of equation (1.1) may be only characterized by anisotropic coefficient \( a(x) \) and singular source \( q \in \overline{\Omega} \). In fact, from Theorems 1.1 and 1.2 it follows that if \( q \) is an isolated local maximum point of \( a(x) \) over \( \overline{\Omega} \) and satisfies \( \langle \nabla a(q), \nu(q) \rangle = 0 \) if \( q \in \partial \Omega \), then equation (1.1) always admits a family of positive solutions \( u_p \), with arbitrarily many interior and boundary spikes accumulating to \( q \) when \( p \) tends to \( +\infty \). Thus as a by-product of our results we find that the presence of anisotropic coefficient \( a(x) \) can lead the concentration point \( q \) of positive solutions to (1.1) to be non-simple (or accumulated) in the sense that there exist more than one (or arbitrarily many) spiky sequences of points converging at the same point.

The proof of our results relies on a very well-known Lyapunov-Schmidt finite dimensional reduction. In Section 2 we exactly describe an approximate solution of equation (1.1). Then we rewrite equation (1.1) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem and an auxiliary nonlinear problem in Section 3. In Section 4 we reduce the problem of finding concentrating solutions of (1.1) to that of finding a critical point of a finite dimensional function and give its asymptotic expansion. In the last section, we give the proofs of Theorems 1.1 and 1.2.

2. Approximating solutions

The basic cells for the construction of an approximate solution of problem (1.1) are given by two standard bubbles

\[
U_\delta(x) = \log \frac{8(1 + \alpha)^2 \delta^2}{(\delta^2 + |x|^{2(1 + \alpha)})^2} \quad \text{and} \quad V_{\delta, \xi}(x) = \log \frac{8\delta^2}{(\delta^2 + |x - \xi|^2)^2},
\]

with \( \alpha \in (-1, +\infty) \setminus \mathbb{N}, \delta > 0 \) and \( \xi \in \mathbb{R}^2 \), which, respectively, are all solutions of the following two equations:

\[
\begin{cases}
- \Delta u = |x|^{2\alpha} e^u & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} |x|^{2\alpha} e^u < +\infty,
\end{cases}
\]

and

\[
\begin{cases}
- \Delta u = e^u & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^u < +\infty,
\end{cases}
\]

(2.2)

(see [5, 6, 31]). Let us define the configuration space in which the concentration points we try to seek belong to

\[
\mathcal{O}_p(q) := \left\{ \xi = (\xi_1, \ldots, \xi_m) \in (B_2(q) \cap \Omega)^l \times (B_2(q) \cap \partial \Omega)^{m-l} \mid \min_{i=1, \ldots, m} (|\xi_i - q| > \frac{1}{p^\kappa}) \right\},
\]

(2.3)

where \( d \geq 0 \) is a sufficiently small but fixed number independent of \( p, l = m \) if \( q \in \Omega \) while \( l = 0, 1, \ldots, m \) if \( q \in \partial \Omega \), and \( \kappa \) is given by

\[
\kappa = 2(m + 1 + \alpha)^2.
\]

Let us fix \( m \in \mathbb{N}, q \in \overline{\Omega} \) and \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \). For numbers \( \mu_0 > 0 \) and \( \mu_i > 0, i = 1, \ldots, m \), yet to be chosen, we set

\[
\gamma = \rho_0^{\frac{1}{2(1 + \alpha)}}, \quad \varepsilon = e^{-\frac{1}{4p}}, \quad \rho_0 = \varepsilon^{\frac{1}{2(1 + \alpha)}}, \quad \nu_0 = \mu_0^{\frac{1}{2(1 + \alpha)}}, \quad \delta_0 = \rho_0 \nu_0, \quad \delta_i = \varepsilon \mu_i,
\]

(2.5)

and

\[
U_0(x) = \frac{1}{\gamma \mu_0^{\frac{1}{2(p-1)}}} \left[ U_{\delta_0}(x - q) + \frac{1}{p} \omega_1 \left( \frac{x - q}{\delta_0} \right) + \frac{1}{p^2} \omega_2 \left( \frac{x - q}{\delta_0} \right) \right],
\]

(2.6)

and

\[
U_i(x) = \frac{1}{\gamma \mu_i^{\frac{1}{2(p-1)}}} \left[ V_{\delta_i, \xi_i}(x) + \frac{1}{p} \tilde{\omega}_1 \left( \frac{x - \xi_i}{\delta_i} \right) + \frac{1}{p^2} \tilde{\omega}_2 \left( \frac{x - \xi_i}{\delta_i} \right) \right].
\]

(2.7)

Here, \( \omega_j \) and \( \tilde{\omega}_j \), \( j = 1, 2 \), respectively, are radial solutions of

\[
\Delta \omega_j + \frac{8(1 + \alpha)^2 |z|^{2\alpha}}{(1 + |z|^{2(1 + \alpha)})^2} \omega_j = \frac{8(1 + \alpha)^2 |z|^{2\alpha}}{(1 + |z|^{2(1 + \alpha)})^2} f_j(|z|) \quad \text{in } \mathbb{R}^2,
\]

(2.8)
and

\[ \Delta \tilde{\omega}_j + \frac{8}{(1 + |z|^2)^2} \tilde{\omega}_j = \frac{8}{(1 + |z|^2)^2} \tilde{f}_j(|z|) \quad \text{in } \mathbb{R}^2, \tag{2.9} \]

where

\[ f_1 = \frac{1}{2} U_1^2, \quad f_2 = \omega_1 U_1 - \frac{1}{3} U_1^3 - \frac{1}{2} \omega_1^2 - \frac{1}{8} U_1^4 + \frac{1}{2} \omega_1 U_1^2, \tag{2.10} \]

and

\[ \tilde{f}_1 = \frac{1}{2} V_{1,0}^2, \quad \tilde{f}_2 = \tilde{\omega}_1 V_{1,0} - \frac{1}{3} V_{1,0}^3 - \frac{1}{2} \tilde{\omega}_1^2 - \frac{1}{8} V_{1,0}^4 + \frac{1}{2} \tilde{\omega}_1 V_{1,0}^2. \tag{2.11} \]

Furthermore, by \cite{4, 14, 15} it follows that for each \( j = 1, 2 \) and \( r = |z| \), as \( r \to +\infty \),

\[ \omega_j(r) = \frac{C_j}{2(1 + \alpha)} \log \left( 1 + r^{2(1 + \alpha)} \right) + O \left( \frac{1}{1 + r^{1 + \alpha}} \right), \quad \partial_r \omega_j(r) = \frac{C_j r^{1 + 2\alpha}}{1 + r^{2(1 + \alpha)}} + O \left( \frac{1}{1 + r^{2 + \alpha}} \right), \tag{2.12} \]

and

\[ \tilde{\omega}_j(r) = \frac{\tilde{C}_j}{2} \log(1 + r^2) + O \left( \frac{1}{1 + r^2} \right), \quad \partial_r \tilde{\omega}_j(r) = \frac{\tilde{C}_j r}{1 + r^2} + O \left( \frac{1}{1 + r^2} \right), \tag{2.13} \]

where

\[ C_j = 8(1 + \alpha)^2 \int_0^{\infty} t^{1 + 2\alpha} \frac{t^{2(1 + \alpha)} - 1}{(t^{2(1 + \alpha)} + 1)^2} f_j(t) \, dt, \quad \tilde{C}_j = 8 \int_0^{\infty} t^{1 - 2} \frac{t^2 - 1}{(t^2 + 1)^2} \tilde{f}_j(t) \, dt. \tag{2.14} \]

In particular,

\[ C_1 = 12(1 + \alpha) - 4(1 + \alpha) \log 8(1 + \alpha)^2, \quad \tilde{C}_1 = 12 - 4 \log 8, \tag{2.15} \]

and

\[ \omega_1(z) = \frac{1}{2} U_1^2(z) + 6 \log \left( |z|^{2(1 + \alpha)} + 1 \right) + \frac{2 \log 8(1 + \alpha)^2 - 10}{|z|^{2(1 + \alpha)} + 1} + \frac{|z|^{2(1 + \alpha)} - 1}{|z|^{2(1 + \alpha)} + 1} \times \left\{ -\frac{1}{2} \log^2 8(1 + \alpha)^2 + 2 \log^2 \left( |z|^{2(1 + \alpha)} + 1 \right) + 4 \int_{|z|^{2(1 + \alpha)}}^{\infty} \frac{ds}{s + 1} \log \frac{s + 1}{s} - 8(1 + \alpha) \log |z| \log \left( |z|^{2(1 + \alpha)} + 1 \right) \right\}, \tag{2.16} \]

and

\[ \tilde{\omega}_1(z) = \frac{1}{2} V_{1,0}^2(z) + 6 \log |z|^2 + 1 + \frac{2 \log 8 - 10}{|z|^2 + 1} + \frac{|z|^2 - 1}{|z|^2 + 1} \times \left\{ -\frac{1}{2} \log^2 8 + 2 \log^2 \left( |z|^2 + 1 \right) + 4 \int_{|z|^2}^{\infty} \frac{ds}{s + 1} \log \frac{s + 1}{s} - 8 \log |z| \log \left( |z|^2 + 1 \right) \right\}. \tag{2.17} \]

We define the approximate solution of problem (1.1) as

\[ U_\xi(x) := \sum_{i=0}^{m} P U_i(x) = \sum_{i=0}^{m} \left[ U_i(x) + H_i(x) \right], \tag{2.18} \]

where \( H_i \) is a correction term defined as the solution of

\[ \begin{cases} -\Delta a H_i + H_i = \nabla \log a(x) \nabla U_i - U_i & \text{in } \Omega, \\ \frac{\partial H_i}{\partial \nu} = -\frac{\partial U_i}{\partial \nu} & \text{on } \partial \Omega. \end{cases} \tag{2.19} \]

In order to state the asymptotic expansion of the functions \( H_i \) in terms of \( \xi_i, \delta_i \) and \( p > 1 \) large enough, we first use the convention

\[ c_0 = \begin{cases} 8\pi (1 + \alpha), & \text{if } q \in \Omega, \\ 4\pi (1 + \alpha), & \text{if } q \in \partial \Omega, \end{cases} \quad c_i = \begin{cases} 8\pi, & \text{if } i = 1, \ldots, l, \\ 4\pi, & \text{if } i = l + 1, \ldots, m. \end{cases} \tag{2.20} \]

Then we have the following Lemma whose proof is listed in the Appendix B.
Lemma 2.1. For any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \) and any \( 0 < \beta < 1 \), then we have that

\[
H_0(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4(1 + \alpha)p} - \frac{C_2}{4(1 + \alpha)p^2} \right) c_0 H(x, q) - \log \left( 8(1 + \alpha)^2 \delta_0^{2(1 + \alpha)} \right) \right.
+ \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \delta_0 + O \left( \delta_0^{\beta/2} \right),
\]

and for each \( i = 1, \ldots, m \),

\[
H_i(x) = \frac{1}{\gamma \mu_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \left[ \left( 1 - \frac{\tilde{C}_1}{4p} - \frac{\tilde{C}_2}{4p^2} \right) c_i H(x, \xi_i) - \log(8\delta_i^2) + \left( \frac{\tilde{C}_1}{p} + \frac{\tilde{C}_2}{p^2} \right) \log \delta_i \right. 
+ O \left( \delta_i^{\beta/2} \right),
\]

uniformly in \( \overline{\Omega} \), where \( H \) is the regular part of Green’s function defined in (1.9) and for each \( j = 1, 2, C_j \) and \( \tilde{C}_j \) are the constants defined in (2.14).

From Lemma 2.1 we can easily check that away from singular source \( q \) and each point \( \xi_i \), namely \( |x - q| \geq 1/p^{2\alpha} \) and \( |x - \xi_i| \geq 1/p^{2\alpha} \) for each \( i = 1, \ldots, m \),

\[
U_\xi(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ \left( 1 - \frac{C_1}{4(1 + \alpha)p} - \frac{C_2}{4(1 + \alpha)p^2} \right) c_0 G(x, q) + O \left( p^{2\alpha(1+\alpha)-1} \delta_0^{1+\alpha} + \delta_0^{\beta/2} \right) \right]
+ \sum_{i=1}^m \frac{1}{\gamma \mu_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \left[ \left( 1 - \frac{\tilde{C}_1}{4p} - \frac{\tilde{C}_2}{4p^2} \right) c_i G(x, \xi_i) + O \left( \delta_i^{\beta/2} \right) \right].
\]

If \( |x - q| < 1/p^{2\alpha} \), by using (2.1), (2.6), (2.21) and the fact that \( H(\cdot, q) \in C^\beta(\overline{\Omega}) \) for any \( \beta \in (0, 1) \) we obtain

\[
PU_0(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ U_1 \left( \frac{x - q}{\delta_0} \right) + \frac{1}{p} \omega_1 \left( \frac{x - q}{\delta_0} \right) + \frac{1}{p^2} \omega_2 \left( \frac{x - q}{\delta_0} \right) + \left( 1 - \frac{C_1}{4(1 + \alpha)p} - \frac{C_2}{4(1 + \alpha)p^2} \right) c_0 H(q, q) \right]
- \log \left( 8(1 + \alpha)^2 \delta_0^{2(1+\alpha)} \right) + \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \delta_0 + O \left( |x - q|^\beta + \delta_0^{\beta/2} \right),
\]

and for any \( k \neq 0 \), by (2.1), (2.7), (2.13) and (2.22),

\[
PU_k(x) = \frac{1}{\gamma \mu_k^{2/(p-1)} |\xi_k - q|^{2\alpha/(p-1)}} \left[ \log \left( \delta_k^{\beta/2} \right) + \frac{1}{p} \omega_1 \left( \frac{x - \xi_k}{\delta_k} \right) + \frac{1}{p^2} \omega_2 \left( \frac{x - \xi_k}{\delta_k} \right) \right]
+ \left( 1 - \frac{\tilde{C}_1}{4p} - \frac{\tilde{C}_2}{4p^2} \right) c_k H(x, \xi_k) - \log(8\delta_k^2) + \left( \frac{\tilde{C}_1}{p} + \frac{\tilde{C}_2}{p^2} \right) \log \delta_k + O \left( \delta_k^{\beta/2} \right)
= \frac{1}{\gamma \mu_k^{2/(p-1)} |\xi_k - q|^{2\alpha/(p-1)}} \left[ \left( 1 - \frac{\tilde{C}_1}{4p} - \frac{\tilde{C}_2}{4p^2} \right) c_k G(q, \xi_k) + O \left( |x - q|^\beta + \delta_k^{\beta/2} \right) \right].
\]

Hence for \( |x - q| < 1/p^{2\alpha} \), by (2.5),

\[
U_\xi(x) = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left[ p + U_1 \left( \frac{x - q}{\delta_0} \right) + \frac{1}{p} \omega_1 \left( \frac{x - q}{\delta_0} \right) + \frac{1}{p^2} \omega_2 \left( \frac{x - q}{\delta_0} \right) + O \left( |x - q|^\beta + \sum_{k=0}^m \delta_k^{\beta/2} \right) \right].
\]
is an appropriate approximation for a solution of problem (1.1) near singular source \(q\) provided that the concentration parameter \(\mu_0\) satisfies the nonlinear relation

\[
\log \left(8(1+\alpha)^2 \mu_0^4\right) = \left(1 - \frac{C_1}{4(1+\alpha)p} - \frac{C_2}{4(1+\alpha)p^2}\right) c_0 H(q, q) + \left(\frac{C_1}{p} + \frac{C_2}{p^2}\right) \log \delta_0
\]

\[
+ \left(1 - \frac{\bar{C}_1}{4p} - \frac{\bar{C}_2}{4p^2}\right) \sum_{k=1}^{m} \frac{\mu_0^{2/(p-1)} |\xi_k - q|^{2\alpha/(p-1)} c_k G(q, \xi_k)}{\mu_k^{2/(p-1)} |\xi_k - q|^{2\alpha/(p-1)} c_k G(q, \xi_k)} \tag{2.25}
\]

Similarly, while if \(|x - \xi_i| < 1/p^{2\alpha}\) with some \(i \in \{1, \ldots, m\},\)

\[
U_{\xi}(\xi) = \frac{1}{\gamma \mu_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \left[ p + V_{1,0} \left(\frac{x - \xi_i}{\delta_i}\right) + \frac{1}{p^2} \omega_1 \left(\frac{x - \xi_i}{\delta_i}\right) + \frac{1}{p^2} \omega_2 \left(\frac{x - \xi_i}{\delta_i}\right) \right] + O \left( |x - \xi_i|^p + p^{2\alpha(1+\alpha)-1} \delta_0^{1+\alpha} + \sum_{k=0}^{m} \delta_k^{\beta/2} \right) \tag{2.26}
\]

is an appropriate approximation for a solution of problem (1.1) near the point \(\xi_i\) provided that for each \(i = 1, \ldots, m,\) the concentration parameter \(\mu_i\) satisfies the nonlinear system

\[
\log \left(8 \mu_i^4\right) = \left(1 - \frac{\bar{C}_1}{4p} - \frac{\bar{C}_2}{4p^2}\right) c_i H(\xi_i, \xi_i) + \left(\frac{\bar{C}_1}{p} + \frac{\bar{C}_2}{p^2}\right) \log \delta_i
\]

\[
+ \left(1 - \frac{\bar{C}_1}{4p} - \frac{\bar{C}_2}{4p^2}\right) \sum_{k=1, k \neq i}^{m} \frac{\mu_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)} c_k G(\xi_i, \xi_k)}{\mu_k^{2/(p-1)} |\xi_k - q|^{2\alpha/(p-1)} c_k G(\xi_k, \xi_k)} \tag{2.27}
\]

Indeed, the parameters \(\mu = (\mu_0, \mu_1, \ldots, \mu_m)\) are well defined in systems (2.25) and (2.27) under the certain region, which is stated in the following lemma and whose proof is postponed in the Appendix B.

**Lemma 2.2.** For any points \(\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q)\) and any \(p > 1\) large enough, systems (2.25) and (2.27) have a unique solution \(\mu = (\mu_0, \mu_1, \ldots, \mu_m)\) satisfying

\[
1/C \leq \mu_i \leq C p^C \quad \text{and} \quad |D_{\xi} \log \mu_i| \leq C p^\kappa, \quad \forall i = 0, 1, \ldots, m, \tag{2.28}
\]

for some \(C > 0.\) Moreover,

\[
\mu_0 = \mu_0(p, \xi) \equiv e^{-\frac{\bar{C}_1}{4p} c_0 H(q, q) + \frac{1}{2} \sum_{k=1}^{m} c_k G(q, \xi_k)} \left[ 1 + O \left( \frac{\log^2 p}{p} \right) \right], \tag{2.29}
\]

and for any \(i = 1, \ldots, m,\)

\[
\mu_i = \mu_i(p, \xi) \equiv e^{-\frac{\bar{C}_1}{4p} c_i H(\xi_i, \xi_i) + \frac{1}{2} c_i G(\xi_i, q) + \frac{1}{2} \sum_{k=1, k \neq i}^{m} c_k G(\xi_i, \xi_k)} \left[ 1 + O \left( \frac{\log^2 p}{p} \right) \right]. \tag{2.30}
\]

**Remark 2.3.** For any \(p > 1\) large enough, we see that if \(|x - q| = \delta_0 |z| < 1/p^2\), by (2.1), (2.5), (2.12) and (2.28),

\[
p + U_1(|z|) + \frac{1}{p^2} \omega_1(|z|) + \frac{1}{p^2} \omega_2(|z|) = p - 2 \log \left(1 + |z|^{2(1+\alpha)}\right) + O(1)
\]

\[
\geq 8\kappa(1+\alpha) \log p + 4 \log \mu_0 + O(1) > 7\kappa(1+\alpha) \log p,
\]
which, together with (2.24), easily implies that \( 0 < U_\xi \leq 2\sqrt{e} \) in \( B_{1/p^{2\alpha}}(q) \), and \( \sup_{B_{1/p^{2\alpha}}(q)} U_\xi \to \sqrt{e} \) as \( p \to +\infty \). While if \( |x - \xi_i| = \delta_i |\tilde{z}| < 1/p^{2\alpha} \) for each \( i = 1, \ldots, m \), by (2.1), (2.5), (2.13) and (2.28),

\[
p + V_{1,0}(|\tilde{z}|) + \frac{1}{p} \tilde{\omega}_1(|\tilde{z}|) + \frac{1}{p^2} \tilde{\omega}_2(|\tilde{z}|) = p - 2 \log (1 + |\tilde{z}|^2) + O(1) \geq 8k \log p + 4 \log \mu + O(1) > 7k \log p.
\]

This, together with (2.3) and (2.26), implies that \( 0 < U_\xi \leq 2\sqrt{e} \) in \( B_{1/p^{2\alpha}}(\xi_i) \), and \( \sup_{B_{1/p^{2\alpha}}(\xi_i)} U_\xi \to \sqrt{e} \) as \( p \to +\infty \). Notice that by the maximum principle, it follows that for any \( i = 1, \ldots, m \), \( G(x, \xi_i) > 0 \) and \( G(x, q) > 0 \) over \( \overline{\Omega} \), and further by (2.23), \( U_\xi \) is a positive, uniformly bounded function over \( \overline{\Omega} \). More precisely,

\[
0 < U_\xi(y) \leq 2\sqrt{e}, \quad \forall \ y \in \overline{\Omega}.
\]

Consider that the scaling of solution to problem (1.1) is as follows:

\[
v(y) = \varepsilon^{2/(p-1)} u(\varepsilon y), \quad \forall \ y \in \overline{\Omega}_p,
\]

where \( \Omega_p := (\varepsilon^{p/4}) \Omega = (1/\varepsilon) \Omega \), then the function \( v(y) \) satisfies

\[
\begin{cases}
- \Delta a(\varepsilon y)v + \varepsilon^2 v = |\varepsilon y - q|^{2\alpha} V^p, & v > 0 \quad \text{in} \quad \Omega_p, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \partial \Omega_p.
\end{cases}
\]

We write \( q' = q/\varepsilon \) and \( \xi_i = \xi / \varepsilon, \ i = 1, \ldots, m \) and define the initial approximate solution of (2.33) as

\[
V_{\xi_i}(y) = \varepsilon^{2/(p-1)} U_\xi(\varepsilon y),
\]

with \( \xi' = (\xi_1', \ldots, \xi_m') \) and \( U_\xi \) defined in (2.18). Let us set

\[
S_p(v) = -\Delta a(\varepsilon y)v + \varepsilon^2 v - |\varepsilon y - q|^{2\alpha} V^p_+, \quad \text{where} \quad v_+ = \max\{v, 0\},
\]

and we consider the functional

\[
I_p(v) = \frac{1}{2} \int_{\Omega_p} a(\varepsilon y) \left( |\nabla v|^2 + \varepsilon^2 v^2 \right) dy - \frac{1}{p+1} \int_{\Omega_p} a(\varepsilon y) |\varepsilon y - q|^{2\alpha} V^{p+1}_+ dy, \quad v \in H^1(\Omega_p),
\]

whose nontrivial critical points are solutions of problem (2.33). Indeed, by virtue of the Hardy and Sobolev embedding inequalities in [16, 37] we have that for all \( \alpha > -1 \), the functional \( I_p \) is well defined in \( H^1(\Omega_p) \). Moreover, by the maximum principle, it is easy to see that problem (2.33) is equivalent to

\[
S_p(v) = 0, \quad v_+ \neq 0 \quad \text{in} \quad \Omega_p, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega_p.
\]

We will seek solutions of problem (2.33) in the form \( v = V_{\xi'} + \phi \), where \( \phi \) will represent a higher-order correction in the expansion of \( v \). Observe that

\[
S_p(V_{\xi'} + \phi) = \mathcal{L}(\phi) + R_{\xi'} + N(\phi) = 0,
\]

where

\[
\mathcal{L}(\phi) = -\Delta a(\varepsilon y) \phi + \varepsilon^2 \phi - W_{\xi'} \phi \quad \text{with} \quad W_{\xi'} = p|\varepsilon y - q|^{2\alpha} V^{p-1}_{\xi'},
\]

and

\[
R_{\xi'} = -\Delta a(\varepsilon y) V_{\xi'} + \varepsilon^2 V_{\xi'} - |\varepsilon y - q|^{2\alpha} V^p_{\xi'}, \quad N(\phi) = -|\varepsilon y - q|^{2\alpha} [(V_{\xi'} + \phi)^p_+ - V^p_{\xi'} - pV^{p-1}_{\xi'} \phi].
\]

In terms of \( \phi \), problem (2.33) becomes

\[
\begin{cases}
\mathcal{L}(\phi) = - \left[ R_{\xi'} + N(\phi) \right] & \text{in} \quad \Omega_p, \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on} \quad \partial \Omega_p.
\end{cases}
\]
For any $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q)$ and $h \in L^\infty(\Omega_p)$, we define a $L^\infty$-norm $\|h\|_* := \sup_{y \in \Omega_p} |H_\xi(y)h(y)|$ with the weight function

$$H_\xi(y) = \left[ \varepsilon^2 + \left( \frac{\varepsilon}{\rho \varepsilon_0} \right)^2 \frac{|\varepsilon_y - \rho \varepsilon_0|^{2\alpha}}{(1 + |\varepsilon_y - \rho \varepsilon_0|)^{4+2\alpha+2\alpha}} + \sum_{i=1}^{m} \frac{1}{\mu_i^2} \left( \frac{1}{1 + \frac{|\varepsilon_y - \xi_i|}{\mu_i}} \right)^{4+2\alpha} \right]^{-1}, \quad (2.39)$$

where $\alpha + 1$ is a sufficiently small but fixed positive number, independent of $p$, such that $-1 < \alpha < \min \{ \alpha, -2/3 \}$. With respect to the $\| \cdot \|_*$-norm, we have the following.

**Proposition 2.4.** Let $m$ be a non-negative integer. There exist constants $C > 0$, $D_0 > 0$ and $p_m > 1$ such that

$$\| R_\xi \|_* \leq \frac{C}{p^2}, \quad (2.40)$$

and

$$W_{\xi} (y) \leq D_0 \left[ \left( \frac{\varepsilon}{\rho \varepsilon_0} \right)^2 \frac{|\varepsilon_y - \rho \varepsilon_0|^{2\alpha}}{(1 + |\varepsilon_y - \rho \varepsilon_0|)^{4+2\alpha+2\alpha}} + \sum_{i=1}^{m} \frac{1}{\mu_i^2} e^{\mathcal{V}_{\xi,i}(\frac{\varepsilon_y - \xi_i}{\mu_i})} \right], \quad (2.41)$$

for any $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q)$ and any $p > p_m$. Moreover, if $|\varepsilon_y - \xi_i| \leq \sqrt{\delta_i}/p^{2\alpha}$ for each $i = 1, \ldots, m$,

$$W_{\xi} (y) = \frac{1}{\mu_i^2} \left( \frac{8}{1 + |\varepsilon_y - \xi_i|^2} \right)^2 \left[ 1 + \frac{1}{p} \left( \omega_1 - U_1, U_1 - \frac{1}{2} U_1^2 \right) + O \left( \frac{\log^4 (\frac{\varepsilon_y - \pi}{\rho \varepsilon_0} + 2)}{p^2} \right) \right], \quad (2.42)$$

while if $|\varepsilon_y - q| \leq \sqrt{\delta_i}/p^{2\alpha}$,

$$W_{\xi} (y) = \left( \frac{\varepsilon}{\rho \varepsilon_0} \right)^2 \frac{8(1+\alpha)^2 |\varepsilon_y - \rho \varepsilon_0|^{2\alpha}}{(1 + |\varepsilon_y - \rho \varepsilon_0|^{2(1+\alpha)})^2} \left[ 1 + \frac{1}{p} \left( \omega_1 - U_1, U_1 - \frac{1}{2} U_1^2 \right) + O \left( \frac{\log^4 (\frac{\varepsilon_y - \pi}{\rho \varepsilon_0} + 2)}{p^2} \right) \right]. \quad (2.43)$$

**Proof.** From (2.18), (2.19) and (2.34) it follows that

$$-\Delta_{\alpha,\varepsilon} V_{\xi} + \varepsilon^2 V_{\xi} = \varepsilon^2 \sum_{i=0}^{m} \sum_{j=0}^{(p-1)} \left( \delta_i |U_i + H_i| \right) = -\Delta_{\alpha,\varepsilon} (U_i + H_i) = -\Delta_{\alpha,\varepsilon} \mathcal{U}_i.$$  

Then by (2.1) and (2.5)-(2.9),

$$-\Delta_{\alpha,\varepsilon} V_{\xi} + \varepsilon^2 V_{\xi} = \left( \frac{\varepsilon}{\rho \varepsilon_0} \right)^2 \frac{|\varepsilon_y - \rho \varepsilon_0|^{2\alpha} e^{\mathcal{V}_{\xi,i}(z)}}{p/(p-1)^{\alpha} \mu_i^{2/(p-1)}} \left( 1 - \frac{1}{p} f_1 - \frac{1}{p^2} f_2 + \frac{1}{p} \omega_1 + \frac{1}{p^2} \omega_2 \right) (z) \quad (2.44)$$

with $z = (\varepsilon_y - q)/\delta_0$ and $\tilde{z} = (\varepsilon_y - \xi_i)/\delta_i$. If $|\varepsilon_y - q| = \delta_0 |z| \geq 1/p^{2\alpha}$ and $|\varepsilon_y - \xi_i| = \delta_i |\tilde{z}| \geq 1/p^{2\alpha}$ for each $i = 1, \ldots, m$, by (2.1), (2.12), (2.13) and (2.28) we can compute that

$$U_i(z) = -p + O (\log p), \quad V_{\xi,i}(\tilde{z}) = -p + O (\log p), \quad \omega_j(z) = \frac{C_{ij} p}{4(1+\alpha)} + O (\log p), \quad \tilde{\omega}_j(\tilde{z}) = \frac{C_{ij} p}{4} + O (\log p)$$

with $j = 1, 2$, and thus, by (2.10), (2.11) and (2.44),

$$-\Delta_{\alpha,\varepsilon} V_{\xi} + \varepsilon^2 V_{\xi} = \left( \frac{\varepsilon}{\rho \varepsilon_0} \right)^2 \frac{|\varepsilon_y - \rho \varepsilon_0|^{2\alpha} e^{\mathcal{V}_{\xi,i}(z)}}{p/(p-1)^{\alpha} \mu_i^{2/(p-1)}} O (p^2) \quad (2.45)$$

In the same region, by (2.3), (2.5), (2.23), (2.28) and (2.34) we obtain

$$|\varepsilon_y - q|^{2\alpha} p^\alpha (y) = O \left( \frac{|\varepsilon_y - q|^{2\alpha}}{p^\alpha} \left( \frac{\log p}{p} \right)^p \right), \quad (2.46)$$
which combined with (2.37), (2.39) and (2.45) easily yields that
\[
|H^p_\xi(y)R^{\cdot}_\xi(y)| \leq C_p \left[ |\xi_y - q|^{2\alpha - 2\alpha \beta} \sum_{i=1}^m \left| y - \frac{\chi_i}{\mu_i} \right|^{2\hat{\alpha}} + \left| \frac{\xi_y - q}{p^2} \left( \frac{\sqrt{\log p}}{p} \right) \right|^p \right]
= o \left( \max \left\{ e^{p(\delta - \alpha)/4(1 + \alpha)}, e^{p\delta / 4}, e^{-p/4} \right\} \right).
\] (2.47)

On the other hand, if \(|\xi_y - \xi_i| = \delta_i|\hat{z}| < 1/p^{2\kappa}\) for some \(i \in \{1, \ldots, m\}\), then, by (2.26), (2.34) and the relation
\[
\left( \frac{p\epsilon^{2/(p-1)}}{\gamma \mu_i^{2/(p-1)}} \right)^p = \frac{1}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}},
\] (2.48)
we find
\[
|\xi_y - q|^{2\alpha V^p_\xi(y)} = \frac{|\xi_y - q|^{2\alpha}}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}} \left\{ 1 + \frac{1}{p} V_{1,0}(\hat{z}) + \frac{1}{p^2} \tilde{\omega}_1(\hat{z}) \right. \right.
+ \left. \left. \frac{1}{p^3} \left[ \tilde{\omega}_2(\hat{z}) + O \left( \left( \frac{p^2}{p^3} |\hat{z}|^\beta + p^2 \sum_{k=0}^m \delta_k^{\beta/2} \right) \right) \right] \right\}^p
\] (2.49)
with \(0 < \beta < \min\{1, 2(1 + \alpha)\}\). From a Taylor expansion of the exponential and logarithmic functions
\[
\left( 1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} \right)^p = e^a \left[ 1 + \frac{1}{p^2} \left( b - \frac{a^2}{2} \right) + \frac{1}{p^3} \left( c - ab + \frac{a^3}{3} + \frac{b^2}{2} - \frac{a^2 b}{2} + \frac{a^4}{8} \right) \right.
+ O \left( \frac{\log^6(|\hat{z}| + 2)}{p^3} \right),
\] (2.50)
which holds for \(|\hat{z}| \leq C p^{p/8}\) provided \(-4 \log(|\hat{z}| + 2) \leq a(\hat{z}) \leq C\) and \(|b(\hat{z})| + |c(\hat{z})| \leq C \log(|\hat{z}| + 2)\), so we can compute that for \(|\xi_y - \xi_i| = \delta_i|\hat{z}| \leq \sqrt{\delta_i}/p^{2\kappa}\),
\[
|\xi_y - q|^{2\alpha V^p_\xi(y)} = \frac{|\xi_y - q|^{2\alpha}}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}} \left[ 1 + \frac{1}{p} \left( \tilde{\omega}_1 - \frac{1}{2} V_{1,0} \right)(\hat{z}) + \frac{1}{p^2} \left( \tilde{\omega}_2 - \tilde{\omega}_1 V_{1,0} + \frac{1}{3} V_{1,0}^3 \right) \right.
+ \left. \frac{1}{2} \gamma \mu_i \left( \frac{1}{2} \tilde{\omega}_1 V_{1,0} + \frac{1}{8} V_{1,0}^2 \right) \right] + O \left( \frac{\log^6(|\hat{z}| + 2)}{p^3} + \delta_i^\beta |\hat{z}|^\beta + \sum_{k=0}^m \delta_k^{\beta/2} \right),
\] (2.51)
and then, by (2.11), (2.37) and (2.44),
\[
R^{\cdot}_\xi(y) = \frac{e^{V_{1,0}^\prime(\hat{z})}}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}} \left( \frac{\log^6(|\hat{z}| + 2)}{p^3} + \delta_i^\beta |\hat{z}|^\beta + \sum_{k=0}^m \delta_k^{\beta/4} \right).
\] (2.52)
Furthermore, in this region, by (2.39) we obtain
\[
|H^p_\xi(y)R^{\cdot}_\xi(y)| \leq \frac{C}{p^2} \left( \left| \frac{\xi_y - \xi_i}{\mu_i} \right| + 1 \right)^{2\hat{\alpha}} \log^6 \left( \left| \frac{\xi_y - \xi_i}{\mu_i} \right| + 2 \right) = O \left( \frac{1}{p^4} \right).
\] (2.53)
As in the region \(\sqrt{\delta_i}/p^{2\kappa} < |\xi_y - \xi_i| = \delta_i |\hat{z}| < 1/p^{2\kappa}\), by (2.11) and (2.44) we get
\[
-\Delta_{\alpha(\xi_y)}V^{\cdot}_\xi + \epsilon^2 V^{\cdot}_\xi = \frac{e^{V_{1,0}^\prime(\hat{z})}}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}} O (p^2),
\] (2.54)
and by (2.49),
\[
|\xi_y - q|^{2\alpha V^p_\xi(y)} = \frac{|\xi_y - q|^{2\alpha e^{V_{1,0}^\prime(\hat{z})}}}{p^{p/(p-1)} \mu_i^{2p/(p-1)} |\xi_i - q|^{2\alpha p/(p-1)}} O (1),
\] (2.55)
because \((1 + \frac{a}{p})^p \leq e^a\). Then in this region, by (2.23), (2.28), (2.37) and (2.39) we conclude

\[
|\mathbf{H}_\xi(y)R_\xi(y)| \leq Cp \left| \frac{y - \xi}{\mu} \right|^{2\alpha} = o \left( \frac{1}{p^4} \right).
\]

(2.56)

Similar to the above arguments in (2.51)-(2.52) and (2.54)-(2.55), we have that if \(|\varepsilon y - q| = \delta_0|z| \leq \sqrt{\delta_0}/p^{2\kappa}\),

\[
R_\xi(y) = \left( \frac{\varepsilon}{\rho_0 \nu_0} \right)^2 \frac{|z|2\alpha U_1(z)}{p^{\rho/(p-1)} \mu_0^{2/(p-1)}} O \left( \frac{\log^6(|z| + 2)}{p^3} + \delta_0^\beta |z|^\beta + \sum_{k=0}^{m} \delta_k^{3/4} \right),
\]

(2.57)

while if \(\sqrt{\delta_0}/p^{2\kappa} \leq |\varepsilon y - q| = \delta_0|z| \leq 1/p^{2\kappa}\),

\[
R_\xi(y) = \left( \frac{\varepsilon}{\rho_0 \nu_0} \right)^2 \frac{|z|2\alpha U_1(z)}{p^{\rho/(p-1)} \mu_0^{2/(p-1)}} O \left( p^2 \right).
\]

(2.58)

Thus in the region \(|\varepsilon y - q| = \delta_0|z| \leq 1/p^{2\kappa}\), by (2.39),

\[
|\mathbf{H}_\xi(y)R_\xi(y)| \leq C p^{\frac{2\alpha - 2\alpha}{p^2}} \left( \frac{|\varepsilon y - q|}{\rho_0 \nu_0} \right)^{2\alpha - 2\alpha} = O \left( \frac{1}{p^4} \right).
\]

(2.59)

As a consequence, putting (2.47), (2.53), (2.56) and (2.59) together, we obtain estimate (2.40).

Finally, it remains to prove the expansions (2.41)-(2.43) for \(W_\xi(y) = p|\varepsilon y - q|2\alpha V_{\xi_0}^{(p)}\). If \(|\varepsilon y - \xi| = \delta_1|\tilde{z}| < 1/p^{2\kappa}\) for some \(i \in \{1, \ldots, m\}\), then by (2.26), (2.34) and (2.48),

\[
W_\xi(y) = p \left( \frac{\varepsilon^{2/(p-1)} |\varepsilon y - q|^{2\alpha}}{\gamma \mu_0^{2/(p-1)} |\xi - q|^{2\alpha}} \right)^{p-1} \left[ p + V_{1,0}(\tilde{z}) + \frac{\bar{\omega}_1(\tilde{z})}{p} + \frac{\bar{\omega}_2(\tilde{z})}{p^2} + O \left( \frac{\delta^\beta |\tilde{z}|^\beta + \sum_{k=0}^{m} \delta_k^{3/2}}{p^2} \right) \right]^{p-1}
\]

\[
= \frac{1}{\mu_0^2} \left[ 1 + \frac{1}{p} V_{1,0}(\tilde{z}) + \frac{1}{p^2} \bar{\omega}_1(\tilde{z}) + \frac{1}{p^3} \bar{\omega}_2(\tilde{z}) + O \left( \frac{\delta^\beta |\tilde{z}|^\beta + \sum_{k=0}^{m} \delta_k^{3/2}}{p^2} \right) \right]^{p-1},
\]

where again we use the notation \(\tilde{z} = (y - \xi)/\mu_1\). In this region, we find

\[
W_\xi(y) \leq C \frac{e^{V_{1,0}(\tilde{z})}}{\mu_0^2} e^{-V_{1,0}(\tilde{z})/p} = O \left( \frac{1}{\mu_0^2} e^{V_{1,0}(\tilde{z})} \right),
\]

(2.61)

because \((1 + a/p)^{p-1} \leq e^a (p-1)/p\) and \(V_{1,0}(\tilde{z}) \geq -p + O(\log p)\). In particular, by a slight modification of formula (2.50),

\[
\left( 1 + \frac{a}{p} + \frac{b}{p^2} + \frac{c}{p^3} \right)^{p-1} = e^a \left[ 1 + \frac{1}{p} \left( b - a - \frac{a^2}{2} \right) + O \left( \frac{\log^4(|\tilde{z}| + 2)}{p^3} \right) \right],
\]

we obtain that, if \(|\varepsilon y - \xi| = \delta_1|\tilde{z}| \leq \sqrt{\delta_1}/p^{2\kappa}\),

\[
W_\xi(y) = \frac{1}{\mu_0^2} e^{V_{1,0}(\tilde{z})} \left[ 1 + \frac{1}{p} \left( \bar{\omega}_1 - V_{1,0} - \frac{1}{2} V_{1,0}^2 \right)(\tilde{z}) + O \left( \frac{\log^4(|\tilde{z}| + 2)}{p^2} \right) \right].
\]

(2.62)

If \(|\varepsilon y - q| = \delta_0|z| \leq 1/p^{2\kappa}\), by (2.5), (2.24) and (2.34) we obtain

\[
W_\xi(y) = p|\varepsilon y - q|^{2\alpha} \left( \frac{\varepsilon^{2/(p-1)}}{\gamma \mu_0^{2/(p-1)}} \right)^{p-1} \left[ p + U_1(z) + \frac{\bar{\omega}_1(z)}{p} + \frac{\bar{\omega}_2(z)}{p^2} + O \left( \frac{\delta_0^\beta |z|^\beta + \sum_{k=0}^{m} \delta_k^{3/2}}{p^2} \right) \right]^{p-1}
\]

\[
= \left( \frac{\varepsilon}{\rho_0 \nu_0} \right)^2 |z|^{2\alpha} \left[ 1 + \frac{1}{p} U_1(z) + \frac{1}{p^2} \bar{\omega}_1(z) + \frac{1}{p^3} \bar{\omega}_2(z) + \frac{1}{p} O \left( \frac{\delta_0^\beta |z|^\beta + \sum_{k=0}^{m} \delta_k^{3/2}}{p^2} \right) \right]^{p-1},
\]

(2.63)

which, together with the fact that \(U_1(z) \geq -p + O(\log p)\), easily yields that in this region,

\[
W_\xi(y) \leq C \left( \frac{\varepsilon}{\rho_0 \nu_0} \right)^2 |z|^{2\alpha} e^{U_1(z)} e^{-U_1(z)/p} = O \left( \left( \frac{\varepsilon}{\rho_0 \nu_0} \right)^2 |z|^{2\alpha} e^{U_1(z)} \right).
\]

(2.64)
Similar to the argument in (2.62), by (2.63) we can derive that if \( |\epsilon y - q| = \delta_0 |z| \leq \sqrt{\delta_0}/p^{2\kappa}, \)

\[
W_\epsilon(y) = \left( \frac{\epsilon}{\rho_0 v_0} \right)^2 |z|^{2\alpha} e^{U_1(z)} \left[ 1 + \frac{1}{p} \left( \omega_1 - U_1 - \frac{1}{2} U_1^2 \right)(z) + O \left( \frac{\log^4 (|z| + 2)}{p^2} \right) \right]. \tag{2.65}
\]

Additionally, if \( |\epsilon y - q| = \delta_0 |z| > 1/p^{2\kappa} \) and \( |\epsilon y - \xi_i| = \delta_i |z| \geq 1/p^{2\kappa} \) for each \( i = 1, \ldots, m \), then, by (2.4), (2.5), (2.23), (2.28) and (2.34) we have that \( W_\epsilon(y) = O(\epsilon^2 p^C (\log p)^{p-1}) \). This completes the proof. \( \square \)

**Remark 2.5.** As for \( W_\epsilon \), let us remark that if \( |\epsilon y - q| \leq 1/p^{2\kappa}, \)

\[
p|\epsilon y - q|^{2\alpha} \left[ V_\epsilon(y) + O \left( \frac{1}{p^3} \right) \right]^{p-2} \leq C \epsilon |\epsilon y - q|^{2\alpha} \left( \frac{p \epsilon^{2/(p-1)} \gamma_{\mu_0}^{2/(p-1)}}{\gamma_{\mu_1}^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \right)^{p-2} \epsilon^{p-2} V_{1,0} \left( \frac{\epsilon y - q}{\rho_0 v_0} \right) \]

and if \( |\epsilon y - \xi_i| \leq 1/p^{2\kappa} \) for some \( i \in \{1, \ldots, m\}, \)

\[
p|\epsilon y - q|^{2\alpha} \left[ V_\epsilon(y) + O \left( \frac{1}{p^3} \right) \right]^{p-2} \leq C \epsilon |\epsilon y - q|^{2\alpha} \left( \frac{p \epsilon^{2/(p-1)} \gamma_{\mu_0}^{2/(p-1)}}{\gamma_{\mu_1}^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \right)^{p-2} \epsilon^{p-2} V_{1,0} \left( \frac{\epsilon y - q}{\rho_0 v_0} \right) \]

Since these estimates are true if \( |\epsilon y - q| > 1/p^{2\kappa} \) and \( |\epsilon y - \xi_i| > 1/p^{2\kappa} \) for each \( i = 1, \ldots, m \), we find

\[
p|\epsilon y - q|^{2\alpha} \left[ V_\epsilon(y) + O \left( \frac{1}{p^3} \right) \right]^{p-2} \leq C \left[ \left( \frac{\epsilon}{\rho_0 v_0} \right)^2 \frac{\epsilon y - q}{\rho_0 v_0} \right]^{2\alpha} e^{U_1 \left( \frac{\epsilon y - q}{\rho_0 v_0} \right)} + \sum_{i=1}^{m} \frac{1}{\mu_i^2} e^{V_{1,0} \left( \frac{\epsilon y - \xi_i}{\rho_0 v_0} \right)} \right]. \tag{2.66}
\]

### 3. The Linearized Problem and The Nonlinear Problem

In this section we will first solve the following linear problem: given \( h \in C(\overline{\Omega}_p) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \partial \Omega_p \), we find a function \( \phi \in H^2(\Omega_p) \) and scalars \( c_{ij} \in \mathbb{R}, \ i = 1, \ldots, m, \ j = 1, J_i, \) such that

\[
\begin{aligned}
\mathcal{L}(\phi) &= -\Delta_a(\epsilon y) \phi + \epsilon^2 \phi - W_\epsilon \phi = h + \frac{1}{a(\epsilon y)} \sum_{i=1}^{m} \sum_{j=1}^{J_i} c_{ij} \chi_i Z_{ij} \quad \text{in} \quad \Omega_p, \\
\frac{\partial \phi}{\partial \nu} &= 0 \quad \text{on} \quad \partial \Omega_p, \\
\int_{\Omega_p} \chi_i Z_{ij} \phi &= 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, J_i,
\end{aligned}
\tag{3.1}
\]

where \( W_\epsilon = p|\epsilon y - q|^{2\alpha} V_{1,0}^{p-1} \) satisfies (2.41)-(2.43), \( J_i = 1 \) if \( i = l + 1, \ldots, m \) while \( J_i = 2 \) if \( i = 1, \ldots, l \), and \( Z_{ij}, \chi_i \) are defined as follows. Let

\[
Z_0(z) = \frac{1}{|z|^{2(1+\alpha)} + 1}, \quad Z_0(z) = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad Z_j(z) = \frac{z_j}{|z|^2 + 1}, \quad j = 1, 2.
\tag{3.2}
\]

It is well known (see [3, 5, 9, 13, 15]) that

- any bounded solution to

\[
\Delta \phi + \frac{8(1+\alpha)^2 |z|^{2\alpha}}{(1 + |z|^{2(1+\alpha)})^2} \phi = 0 \quad \text{in} \quad \mathbb{R}^2,
\tag{3.3}
\]

where \(-1 < \alpha \notin \mathbb{N}\), is proportional to \( Z_0; \)
any bounded solution to
\[ \Delta \phi + \frac{8(1 + \alpha)^2|z|^{2\alpha}}{(1 + |z|^{2(1+\alpha)})^2}\phi = 0 \quad \text{in } \mathbb{R}^2_+, \quad \frac{\partial \phi}{\partial v} = 0 \quad \text{on } \partial \mathbb{R}^2_+, \] (3.4)
where \(-1 < \alpha \neq \mathbb{N}\) and \(\mathbb{R}^2_+ := \{(z_1, z_2) : z_2 > 0\}\), is proportional to \(Z_q\);

- any bounded solution to
\[ \Delta \phi + \frac{8}{(1 + |z|^{2})^2}\phi = 0 \quad \text{in } \mathbb{R}^2, \] (3.5)
is a linear combination of \(Z_j, j = 0, 1, 2;\)
- any bounded solution to
\[ \Delta \phi + \frac{8}{(1 + |z|^{2})^2}\phi = 0 \quad \text{in } \mathbb{R}^2, \quad \frac{\partial \phi}{\partial v} = 0 \quad \text{on } \partial \mathbb{R}^2_+, \] (3.6)
is a linear combination of \(Z_j, j = 0, 1.\)

Next, let us consider a large but fixed positive number \(R_0\) and smooth non-increasing cut-off function \(\chi : \mathbb{R} \to [0, 1]\) such that \(\chi(r) = 1\) for \(r \leq R_0\), and \(\chi(r) = 0\) for \(r \geq R_0 + 1\).

For the interior spike case, i.e. \(q \in \partial \Omega\) and \(\xi_i \in \partial \Omega\) with \(i = 1, \ldots, l\), we define
\[ \chi_q(y) = \chi \left( \frac{|y - q|}{\mu_i} \right), \quad Z_q(y) = \frac{\varepsilon}{\rho_0 \varepsilon} \chi_q(y) \left( \frac{\varepsilon y - q}{\rho_0 \varepsilon} \right), \] (3.7)
and for any \(i = 1, \ldots, l\) and \(j = 0, 1, 2\),
\[ \chi_i(y) = \chi \left( \frac{|y - \xi_i|}{\mu_i} \right), \quad Z_{ij}(y) = \frac{1}{\mu_i} \chi_i \left( \frac{y - \xi_i}{\mu_i} \right). \] (3.8)

For the boundary spike case, i.e. \(q \in \partial \Omega\) and \(\xi_i \in \partial \Omega\) with \(i = l + 1, \ldots, m\), we need first to straighten the boundary. Namely, at each boundary point \(q\) and \(\xi_i, i = l + 1, \ldots, m\), we define the planar rotation maps \(A_q : \mathbb{R} \to \mathbb{R}^2_+\) and \(A_i : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \(A_q \nu q(0) = \nu_{y i}(0)\) and \(A_i \nu q(0) = \nu_{y l}(0)\). Let \(G(x)\) be the defining function for the boundary \(A_q (\partial \Omega - \{q\})\) or \(A_i (\partial \Omega - \{\xi_i\})\) in a small neighborhood \(B_3(0)\) of the origin, that is, there exist \(R_1 > 0, \delta > 0\) and a small and a smooth function \(\xi : (-R_1, R_1) \to \mathbb{R}\) satisfying \(\xi(0) = 0, \xi'(0) = 0\) and such that \(A_q (\Omega - \{q\}) \cap B_3(0)\) or \(A_i (\Omega - \{\xi_i\}) \cap B_3(0)\) can be rewritten in the form \((x_1, x_2) : -R_1 < x_1 < R_1, x_2 > G(x) \) \(\cap B_3(0)\). Furthermore, we consider the flattening changes of variables \(F_q : A_q (\Omega - \{q\}) \cap B_3(0) \to \mathbb{R}^2_+\) and \(F_i : A_i (\Omega - \{\xi_i\}) \cap B_3(0) \to \mathbb{R}^2_+\), respectively defined by
\[ F_q = (F_{q1}, F_{q2}), \quad \text{where} \quad F_{q1} = x_1 + \frac{x_2 - G(x_1)}{1 + |G'(x_1)|^2} G'(x_1), \quad F_{q2} = x_2 - G(x_1), \] (3.9)
and for any \(i = l + 1, \ldots, m\),
\[ F_i = (F_{i1}, F_{i2}), \quad \text{where} \quad F_{i1} = x_1 + \frac{x_2 - G(x_1)}{1 + |G'(x_1)|^2} G'(x_1), \quad F_{i2} = x_2 - G(x_1). \] (3.10)

Let us denote
\[ F_p^p(y) = e^{p/4} F_q(A_q(e^{-p/4} y - q)), \quad F_i^p(y) = e^{p/4} F_i(A_i(e^{-p/4} y - \xi_i)), \] (3.11)
\[ i = l + 1, \ldots, m. \]
Here, recalling that \(q \in \partial \Omega\), we define
\[ \chi_q(y) = \chi \left( \frac{\varepsilon}{\rho_0 \varepsilon} |F_p^p(y)| \right), \quad Z_q(y) = \frac{\varepsilon}{\rho_0 \varepsilon} Z_q \left( \frac{\varepsilon}{\rho_0 \varepsilon} F_p^p(y) \right), \] (3.12)
and for any \(i = l + 1, \ldots, m\) and \(j = 0, 1\),
\[ \chi_i(y) = \chi \left( \frac{1}{\mu_i} |F_i^p(y)| \right), \quad Z_{ij}(y) = \frac{1}{\mu_i} Z_{ij} \left( \frac{1}{\mu_i} F_i^p(y) \right). \] (3.13)
It is important to note that if \( q \in \partial \Omega \), then the maps \( F^p_q \) and \( F^p_i \), \( i = l + 1, \ldots, m \) preserve the Neumann boundary condition.

**Proposition 3.1.** Let \( q \in \Omega \) and \( m \) be a non-negative integer. Then there exist constants \( C > 0 \) and \( p_m > 1 \) such that for any \( p > p_m \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \) and any \( h \in C(\Omega_p) \), there is a unique solution \( \phi \in H^2(\Omega_p) \) and scalars \( c_{ij} \in \mathbb{R} \), \( i = 1, \ldots, m, \ j = 1, J_i \) to problem (3.1), which satisfy

\[
\|\phi\|_{L^\infty(\Omega_p)} \leq C p \|h\|, \quad \text{and} \quad \sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| \leq C \|h\|_s. \quad (3.15)
\]

We carry out the proof in the following four steps.

**Step 1:** Constructing a suitable barrier.

**Lemma 3.2.** There exist constants \( R_1 > 0 \) and \( C > 0 \), independent of \( p \), such that for any sufficiently large \( p \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \) and any \( \alpha \in (-1, \min\{\alpha, -2/3\}) \), there exists

\[
\psi : \Omega_p \left\{ \bigcup_{i=1}^m B_{R_1 \mu_i}(\xi'_i) \cup B_{R_1 \phi_{y_{0v_0}}}(q') \right\} \rightarrow \mathbb{R}
\]

smooth and positive so that

\[
\mathcal{L}(\psi) \geq \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 + \sum_{i=1}^m \frac{1}{\mu_i^2} \frac{|\xi_{i}' - \xi_i|^{4+2 \alpha}}{4+2 \alpha} + \varepsilon^2 \quad \text{in} \quad \Omega_p \setminus \left( \bigcup_{i=1}^m B_{R_1 \mu_i}(\xi'_i) \cup B_{R_1 \phi_{y_{0v_0}}}(q') \right),
\]

\[
\frac{\partial \psi}{\partial \nu} \geq 0 \quad \text{on} \quad \partial \Omega_p \setminus \left( \bigcup_{i=1}^m B_{R_1 \mu_i}(\xi'_i) \cup B_{R_1 \phi_{y_{0v_0}}}(q') \right),
\]

\[
\psi > 0 \quad \text{in} \quad \Omega_p \setminus \left( \bigcup_{i=1}^m B_{R_1 \mu_i}(\xi'_i) \cup B_{R_1 \phi_{y_{0v_0}}}(q') \right),
\]

\[
\psi \geq 1 \quad \text{on} \quad \Omega_p \cap \left( \bigcup_{i=1}^m \partial B_{R_1 \mu_i}(\xi'_i) \cup \partial B_{R_1 \phi_{y_{0v_0}}}(q') \right).
\]

Moreover, \( \psi \) is uniformly bounded, i.e.

\[
1 < \psi \leq C \quad \text{in} \quad \Omega_p \setminus \left( \bigcup_{i=1}^m B_{R_1 \mu_i}(\xi'_i) \cup B_{R_1 \phi_{y_{0v_0}}}(q') \right).
\]

**Proof.** Let \( \Psi_0(y) = \psi_0(\varepsilon y) \), where \( \psi_0 \) is the solution to

\[
\begin{aligned}
-\Delta_\alpha \psi_0 + \psi_0 &= 1 & & \text{in} \quad \Omega, \\
\frac{\partial \psi_0}{\partial \nu} &= 1 & & \text{on} \quad \partial \Omega.
\end{aligned}
\]

Then

\[
-\Delta_\alpha \psi_0 + \varepsilon^2 \psi_0 = \varepsilon^2 \quad \text{in} \quad \Omega_p, \quad \text{and} \quad \frac{\partial \psi_0}{\partial \nu} = \varepsilon \quad \text{on} \quad \partial \Omega_p.
\]

Obviously, \( \Psi_0 \) is a positive, uniformly bounded function over \( \Omega_p \). Take the function

\[
\psi = \left( 1 - \frac{1}{|\xi_{i}' - \xi_i|^{2(1+\alpha)}} \right) + \sum_{i=1}^m \left( 1 - \frac{1}{|\xi_{i}' - \xi_i|^{2(1+\alpha)}} \right) + C_1 \Psi_0(y).
\]

As a result, it is directly checked that, choosing the positive constant \( C_1 \) larger if necessary, \( \psi \) meets the required conditions of the lemma for numbers \( R_1 \) and \( p \) large enough. \( \square \)
Step 2: Handing a linear equation. Given $h \in C^{0,\alpha}(\Omega_p)$ and $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q)$, we first study the linear equation
\[
\begin{cases}
\mathcal{L}(\phi) = -\Delta_{a(x,y)} \phi + \varepsilon^2 \phi - W_{\xi' \phi} = h & \text{in } \Omega_p, \\
\frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Omega_p.
\end{cases}
\tag{3.16}
\]
For the solution of (3.16) satisfying more orthogonality conditions than those in (3.1), we prove the following a priori estimate.

Lemma 3.3. There exist $R_0 > 0$ and $p_m > 1$ such that for any $p > p_m$ and any solution $\phi$ of (3.16) with the orthogonality conditions
\[
\int_{\Omega_p} \chi_q Z_q \phi = 0 \quad \text{and} \quad \int_{\Omega_p} \chi_i Z_{ij} \phi = 0, \quad i = 1, \ldots, m, \ j = 0, 1, J_i,
\tag{3.17}
\]
we have
\[
\|\phi\|_{L^\infty(\Omega_p)} \leq C\|h\|_*,
\tag{3.18}
\]
where $C > 0$ is independent of $p$.

Proof. Take $R_0 = 2R_1$ with $R_1$ as the constant in the previous step. Since $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q)$, $\rho_0 v_0 = o(1/p^\gamma)$ and $\varepsilon \mu_1 = o(1/p^\gamma)$ for $p$ large enough, we find $B_{R_1, \rho_0 v_0/\varepsilon(q')}$ and $B_{R_1, \mu_1}(\xi_i)$, $i = 1, \ldots, m$ disjointed. Let $h$ be bounded and $\phi$ be a bounded solution to (3.16) satisfying (3.17). Let us consider the “inner norm”
\[
\|\phi\|_{\star \star} = \sup_{y \in \Omega_p \cap \left( \bigcup_{i=1}^{m} B_{R_1, \rho_0 v_0/\varepsilon(q')} \right)} |\phi(y)|,
\tag{3.19}
\]
and claim that there is a constant $C > 0$ independent of $p$ such that
\[
\|\phi\|_{L^\infty(\Omega_p)} \leq C (\|\phi\|_{\star \star} + \|h\|_*) .
\tag{3.20}
\]
We will establish this estimate with the use of the barrier $\psi$ constructed by Lemma 3.2. In fact, we take
\[
\tilde{\phi}(y) = C_1 (\|\phi\|_{\star \star} + \|h\|_*) \psi(y) \quad \forall \ y \in \overline{\Omega}_p \ \setminus \ \left( \bigcup_{i=1}^{m} B_{R_1, \rho_0 v_0/\varepsilon(q')} \right),
\]
where $C_1 > 0$ is a large constant, independent of $p$. Then for $y \in \Omega_p \ \setminus \ \left( \bigcup_{i=1}^{m} B_{R_1, \rho_0 v_0/\varepsilon(q')} \right)$,
\[
\mathcal{L} (\tilde{\phi} \pm \phi) (y) \geq C_1 \|h\|_\star \left\{ \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{1}{\left| y - \bar{y} \right|^q} \right\}^{4+2\alpha} + \sum_{i=1}^{m} \frac{1}{\left| y - \xi_i \right|^{4+2\alpha} + \varepsilon^2} \pm h(y) \geq |h(y)| \pm h(y) \geq 0,
\]
for $y \in \partial \Omega_p \ \setminus \ \left( \bigcup_{i=1}^{m} \partial B_{R_1, \rho_0 v_0/\varepsilon(q')} \right)$,
\[
\frac{\partial}{\partial \nu} (\tilde{\phi} \pm \phi)(y) \geq 0,
\]
and for $y \in \Omega_p \ \cap \ \left( \bigcup_{i=1}^{m} \partial B_{R_1, \rho_0 v_0/\varepsilon(q')} \right)$,
\[
(\tilde{\phi} \pm \phi)(y) > \|\phi\|_{\star \star} \pm \phi(y) \geq |\phi(y)| \pm \phi(y) \geq 0.
\]
By the maximum principle it follows that $-\tilde{\phi} \leq \phi \leq \tilde{\phi}$ on $\overline{\Omega}_p \ \setminus \ \left( \bigcup_{i=1}^{m} B_{R_1, \rho_0 v_0/\varepsilon(q')} \right)$, which easily implies that estimate (3.20) holds.

We prove the lemma by contradiction. Assume that there are sequences of parameters $p_n \to +\infty$, points $\xi^n = (\xi^n_1, \ldots, \xi^n_m) \in \mathcal{O}_{p_n}(q)$, functions $W_{(\xi^n)}$, $h_n$ and associated solutions $\phi_n$ of equation (3.16) with orthogonality conditions (3.17) such that
\[
\|\phi_n\|_{L^\infty(\Omega_{p_n})} = 1 \quad \text{but} \quad \|h_n\|_\star \rightarrow 0 \quad \text{as} \ n \rightarrow +\infty.
\tag{3.21}
\]
For \( q \in \Omega \), we consider \( \hat{\phi}_q^n(z) = \phi_n((\mu_0^n v_0^n z + q)/\varepsilon_n) \) and \( \hat{h}_q^n(z) = h_n((\mu_0^n v_0^n z + q)/\varepsilon_n) \), where \( \mu^n = (\mu_0^n, \mu_1^n, \ldots, \mu_m^n) \), \( \varepsilon_n = \exp \{-\frac{1}{\delta_n^2} \} \), \( \rho_0^n = (\varepsilon_n)^{1/\Omega_n} \) and \( v_0^n = (\mu_0^n)^{1/\Omega_n} \). Observe that

\[
(- \Delta_{a(\varepsilon_n y)} \phi_n + \varepsilon_n^2 \phi - W_{(\xi^n)' y} \phi_n)_{y = \rho_0^n v_0^n z + q} = \begin{cases} 
\varepsilon_n \left( \frac{\rho_0^n v_0^n \varepsilon_n}{\varepsilon_n} \right)^2 
\hat{a}_n(z) \hat{\phi}_q^n + \left( \rho_0^n v_0^n \right)^2 \hat{\phi}_q^n - \left( \rho_0^n v_0^n \varepsilon_n \right)^2 \hat{W}_n \hat{\phi}_q^n \end{cases} (z),
\]

where

\[
\hat{a}_n(z) = a(\rho_0^n v_0^n z + q), \quad \hat{W}_n(z) = W_{(\xi^n)' y}((\rho_0^n v_0^n z + q)/\varepsilon_n).\]

From the expansion of \( W_{(\xi^n)' y} \) in (2.43), it follows that \( \hat{\phi}_q^n(z) \) solves

\[
-\Delta_{a(\varepsilon_n z)} \hat{\phi}_q^n + \left( \rho_0^n v_0^n \right)^2 \hat{\phi}_q^n - \frac{8(1 + \alpha)^2|z|^{2 \alpha}}{(1 + |z|^{2(1+\alpha)})^2} \left[ 1 + O \left( \frac{1}{\varepsilon} \right) \right] \hat{\phi}_q^n(z) = \left( \rho_0^n v_0^n \right)^2 \hat{h}_q^n(z)
\]

for any \( z \in B_{R_0 + \varepsilon}(0) \), Owing to (3.21) and the definition of the \( \| \cdot \|_\ast \)-norm with respect to (2.39), we find that for any \( \theta \in (1, -1/\hat{a}) \), \( (\rho_0^n v_0^n \varepsilon_n)^2 \hat{h}_q^n \to 0 \) in \( L^q(B_{R_0 + \varepsilon}(0)) \). Since \( \frac{8(1 + \alpha)^2|z|^{2 \alpha}}{(1 + |z|^{2(1+\alpha)})^2} \) is bounded in \( L^q(B_{R_0 + \varepsilon}(0)) \), standard elliptic regularity implies that \( \hat{\phi}_q^n \) converges uniformly over compact subsets near the origin to a bounded solution \( \hat{\phi}_q^\infty \) of equation (3.3) with the property

\[
\int_{\mathbb{R}^2} \chi_{\mathbb{R}^2} \hat{\phi}_q^\infty = 0.
\]

Then \( \hat{\phi}_q^\infty \) is proportional to \( Z_q \). Since \( \int_{\mathbb{R}^2} \chi_{\mathbb{R}^2}^2 > 0 \), by (3.22) we have that \( \hat{\phi}_q^\infty \equiv 0 \) in \( B_{R_1}(0) \).

For \( q \in \partial \Omega \), we consider \( \hat{\phi}_q^n(z) = \phi_n((A_q^n)^{-1} \varepsilon_n^{-1} \mu_0^n v_0^n z + q') \), where \( q' = q / \varepsilon_n \) and \( A_q^n : \mathbb{R}^2 \to \mathbb{R}^2 \) is a rotation map such that \( A_q^n \nu_{\Omega_n} = \nu_{\Omega_n} [0) \). Similarly to the above argument, by using the expansion of \( W_{(\xi^n)' y} \) in (2.43) and elliptic regularity we can derive that \( \hat{\phi}_q^n \) converges uniformly over compact sets to a bounded solution \( \hat{\phi}_q^\infty \) of equation (3.4) with the property

\[
\int_{\mathbb{R}^2} \chi_{\mathbb{R}^2} \hat{\phi}_q^\infty = 0.
\]

Then \( \hat{\phi}_q^\infty \) is proportional to \( Z_q \). Since \( \int_{\mathbb{R}^2} \chi_{\mathbb{R}^2}^2 > 0 \), by (3.23) we have that \( \hat{\phi}_q^\infty \equiv 0 \) in \( B_{R_1^+}(0) \).

For each \( i \in \{1, \ldots, l\} \), we have that \( \xi^n_i \in \Omega \) and we consider \( \hat{\phi}_i^n(z) = \phi_n(\mu_i^n z + (\xi^n_i)') \) with \( (\xi^n_i)' = \xi^n_i / \varepsilon_n \). Notice that

\[
h_n(y) = \left(- \Delta_{a(\varepsilon_n y)} \phi_n + \varepsilon_n^2 y - W_{(\xi^n)' y} \phi_n \right)_{y = \mu_i^n z + (\xi^n_i)'} = (\mu_i^n)^{-2} \left[ -\Delta_{a(\varepsilon_n z)} \hat{\phi}_i^n + \varepsilon_n^2 (\mu_i^n)^2 \hat{\phi}_i^n - (\mu_i^n)^2 \hat{W}_n \hat{\phi}_i^n \right] (z),
\]

where

\[
\hat{a}_n(z) = a(\varepsilon_n \mu_i^n z + (\xi^n_i)') \quad \hat{W}_n(z) = W_{(\xi^n)' y}(\mu_i^n z + (\xi^n_i)').
\]

Using the expansion of \( W_{(\xi^n)' y} \) in (2.42) and elliptic regularity, we find that \( \hat{\phi}_i^n \) converges uniformly over compact subsets near the origin to a bounded solution \( \hat{\phi}_i^\infty \) of equation (3.5), which satisfies

\[
\int_{\mathbb{R}^2} \chi_{\mathbb{R}^2} \hat{\phi}_i^\infty = 0 \quad \text{for} \quad j = 0, 1, 2.
\]

Thus \( \hat{\phi}_i^\infty \) is a linear combination of \( Z_j \), \( j = 0, 1, 2 \). But \( \int_{\mathbb{R}^2} \chi_{\mathbb{R}^2}^2 > 0 \) and \( \int_{\mathbb{R}^2} \chi_{\mathbb{R}^2} Z_i = 0 \) for \( j \neq t \). Hence (3.24) implies that \( \hat{\phi}_i^\infty \equiv 0 \) in \( B_{R_1}(0) \).

Finally, for each \( i \in \{l + 1, \ldots, m\} \), we have that \( \xi^n_i \in \partial \Omega \) and we consider \( \hat{\phi}_i^n(z) = \phi_n((A_q^n)^{-1} \mu_0^n z + (\xi^n_i)') \), where \( A_q^n : \mathbb{R}^2 \to \mathbb{R}^2 \) is a rotation map such that \( A_q^n \nu_{\Omega_n} = \nu_{\Omega_n} [0) \). Similar to the above argument, by
Thus \( \hat{\phi}_i^\infty \) and \( \hat{\phi}_j^\infty \) are linear combinations of \( Z_i \), \( j = 0, 1 \). Note that \( \int_{\mathbb{R}^2^+} \chi Z_j^2 > 0 \) and \( \int_{\mathbb{R}^2^+} \chi Z_j Z_t = 0 \) for \( j \neq t \). Hence (3.25) implies that \( \hat{\phi}_i^\infty \equiv 0 \) in \( B_{R_t}^+ (0) \). Furthermore, by (3.19) we obtain \( \lim_{n \to +\infty} \|\phi_n\|_{**} = 0 \). But (3.20) and (3.21) tell us \( \liminf_{n \to +\infty} \|\phi_n\|_{**} > 0 \), which is a contradiction. \( \square \)

**Step 3:** Establishing uniform an a priori estimate for solutions to (3.16) that satisfy orthogonality conditions with respect to \( Z_{ij} \), \( j = 1, J_i \) only.

**Lemma 3.4.** For \( p \) large enough, if \( \phi \) solves (3.16) and satisfies
\[
\int_{\Omega_p} \chi_i Z_{ij} \phi = 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, J_i, \tag{3.26}
\]
then
\[
\|\phi\|_{L^\infty (\Omega_p)} \leq C_p \|h\|_{**}, \tag{3.27}
\]
where \( C > 0 \) is independent of \( p \).

**Proof.** With no loss of generality we prove the validity of estimate (3.27) only under the case \( q \in \partial \Omega \), because for the other case \( q \in \Omega \) this estimate can also be established in an analogous but a little bit more simple consideration.

Fix \( q \in \partial \Omega \) and let \( R > R_0 + 1 \) be large and fixed, \( d > 0 \) small but fixed. We denote that for \( i = 1, \ldots, m, \)
\[
\hat{Z}_q (y) = Z_q (y) - \frac{\varepsilon}{\rho_0 v_0} + a_i G (\varepsilon y, q), \quad \hat{Z}_i (y) = Z_i (y) - \frac{1}{\mu_i} + a_{i0} G (\varepsilon y, \xi_i), \tag{3.28}
\]
where
\[
a_i = \frac{\varepsilon}{\rho_0 v_0 \left[ H (q, q) - \frac{4 (1 + \alpha)}{c_0} \log (\rho_0 v_0 R) \right]} \quad , \quad a_{i0} = \frac{1}{\mu_i [H (\xi_i, \xi_i) - \frac{4}{c_0} \log (\varepsilon \mu_i R)]}. \tag{3.29}
\]
From estimate (2.28), definitions (2.3), (2.5), (3.7), (3.8), (3.13), (3.14), and expansions (A3), (A5) we obtain
\[
C_1 p \leq - \log (\rho_0 v_0 R) \leq C_2 p, \quad C_1 p \leq - \log (\varepsilon \mu_i R) \leq C_2 p, \tag{3.30}
\]
and
\[
\hat{Z}_q (y) = O \left( \frac{\varepsilon G (\varepsilon y, q)}{p \rho_0 v_0} \right), \quad \hat{Z}_i (y) = O \left( \frac{G (\varepsilon y, \xi_i)}{p \mu_i} \right). \tag{3.31}
\]
Let \( \eta_1 \) and \( \eta_2 \) be radial smooth cut-off functions in \( \mathbb{R}^2 \) such that
\[
0 \leq \eta_1 \leq 1; \quad |\nabla \eta_1| \leq C \text{ in } \mathbb{R}^2; \quad \eta_1 \equiv 1 \text{ in } B_R (0); \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R_{+1}} (0); \quad 0 \leq \eta_2 \leq 1; \quad |\nabla \eta_2| \leq C \text{ in } \mathbb{R}^2; \quad \eta_2 \equiv 1 \text{ in } B_{\delta_0} (0); \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{\delta_0} (0).
\]
Set
\[
\eta_{q1} (y) = \eta_1 \left( \frac{\varepsilon}{\rho_0 v_0} \left| F^p_q (y) \right| \right), \quad \eta_{q2} (y) = \eta_2 \left( \varepsilon \left| F^p_q (y) \right| \right), \tag{3.32}
\]
and for any \( i = 1, \ldots, l, \)
\[
\eta_{1i} (y) = \eta_1 \left( \frac{1}{\mu_i} \left| y - \xi_i \right| \right), \quad \eta_{2i} (y) = \eta_2 \left( \varepsilon \left| y - \xi_i \right| \right), \tag{3.33}
\]
and for any \( i = l + 1, \ldots, m, \)
\[
\eta_{1i} (y) = \eta_1 \left( \frac{1}{\mu_i} \left| F^p_i (y) \right| \right), \quad \eta_{2i} (y) = \eta_2 \left( \varepsilon \left| F^p_i (y) \right| \right). \tag{3.34}
\]
Let us define the two test functions
\[ \tilde{Z}_q = \eta q_1 Z_q + (1 - \eta q_1) \eta q_2 \bar{Z}_q, \quad \tilde{Z}_{i0} = \eta_{i1} Z_{i0} + (1 - \eta_{i1}) \eta_{i2} \bar{Z}_{i0}. \] (3.35)

Given \( \phi \) satisfying (3.16) and (3.26), let
\[ \tilde{\phi} = \phi + d_q \tilde{Z}_q + \sum_{i=1}^m d_i \tilde{Z}_{i0} + \sum_{i=1}^m \sum_{j=1}^{J_i} e_{ij} \chi_i Z_{ij}. \] (3.36)

We will first prove the existence of \( d_q, d_i \) and \( e_{ij} \) such that \( \tilde{\phi} \) satisfies the orthogonality conditions in (3.17). Testing (3.36) against \( \chi_i Z_{ij} \) and using the orthogonality conditions in (3.17) and (3.26) for \( j = 1, J_i \) together with the fact that \( \chi_i \chi_k \equiv 0 \) if \( i \neq k \), we find
\[ e_{ij} = \left( -d_q \int_{\Omega_p} \chi_i Z_{ij} \tilde{Z}_q - \sum_{k=1}^m d_k \int_{\Omega_p} \chi_i Z_{ij} \bar{Z}_{k0} \right) / \int_{\Omega_p} \chi_i^2 Z_{ij}^2, \quad i = 1, \ldots, m, \ j = 1, J_i. \] (3.37)

Note that if \( i = 1, \ldots, l \) and \( j = 1, 2 \), by (3.8),
\[ \int_{\Omega_p} \chi_i Z_{ij} \tilde{Z}_{i0} = \int_{\mathbb{R}^2} \chi Z_j Z_0 = 0, \quad \int_{\Omega_p} \chi_i^2 Z_{ij}^2 = \int_{\mathbb{R}^2} \chi^2 Z_j^2 = C_j > 0, \]
while if \( i = l + 1, \ldots, m \) and \( j = 1 \), by (3.10), (3.12) and (3.14),
\[ \int_{\Omega_p} \chi_i Z_{ij} \tilde{Z}_{i0} = \int_{\mathbb{R}^2} \chi Z_j Z_0[1 + O(\varepsilon \mu_i |z|)] = O(\varepsilon \mu_i), \quad \int_{\Omega_p} \chi_i^2 Z_{ij}^2 = \int_{\mathbb{R}^2} \chi^2 Z_j^2[1 + O(\varepsilon \mu_i |z|)] = C_j / 2 + O(\varepsilon \mu_i). \]

By (3.31) and (3.35),
\[ \int_{\Omega_p} \chi_i Z_{ij} \tilde{Z}_q = O\left( \varepsilon \mu_i d_i + \varepsilon \mu_i \log \frac{p}{p_0 v_0} |d_q| + \sum_{k \neq i} \frac{\mu_k \log p}{p \mu_k} |d_k| \right), \quad \int_{\Omega_p} \chi_i Z_{ij} \bar{Z}_{k0} dy = O\left( \frac{\mu_i \log p}{p \mu_k} \right), \quad \forall k \neq i. \]

Then
\[ |e_{ij}| \leq C \left( \varepsilon \mu_i |d_i| + \varepsilon \mu_i \log \frac{p}{p_0 v_0} |d_q| + \sum_{k \neq i} \frac{\mu_k \log p}{p \mu_k} |d_k| \right). \] (3.38)

So we only need to consider \( d_q \) and \( d_i \). Multiplying definition (3.36) by \( \chi_q Z_q \) and \( \chi_i Z_{i0} \), respectively, and using the orthogonality conditions in (3.17) for \( q \) and \( j = 0 \) and the fact that \( \chi_q \chi_k \equiv 0 \) for all \( k \), we obtain a system of \((d_q, d_1, \ldots, d_m)\)
\[ d_q \int_{\Omega_p} \chi_q Z_q \tilde{Z}_q = - \int_{\Omega_p} \chi_q Z_q \tilde{Z}_{k0}, \quad \int_{\Omega_p} \chi_i Z_{i0} \tilde{Z}_q = - \int_{\Omega_p} \chi_i Z_{i0} \tilde{Z}_{k0}, \quad i = 1, \ldots, m. \] (3.39)

By (3.13), (3.14), (3.31) and (3.35) we can compute
\[ \int_{\Omega_p} \chi_q Z_q \tilde{Z}_q = \int_{\mathbb{R}^2} \chi Z_q^2[1 + O(\rho_0 v_0 |z|)] = C_q / 2 + O(\rho_0 v_0), \quad \int_{\Omega_p} \chi_q Z_q \bar{Z}_{k0} = O\left( \frac{\rho_0 v_0 \log p}{p \varepsilon \mu_k} \right), \]
and
\[ \int_{\Omega_p} \chi_i Z_{i0} \tilde{Z}_q = O\left( \frac{\varepsilon \mu_i \log p}{p_0 v_0} \right), \quad \int_{\Omega_p} \chi_i Z_{i0} \bar{Z}_{k0} = O\left( \frac{\mu_i \log p}{p \mu_k} \right), \quad i \neq k. \]

Moreover, if \( i = 1, \ldots, l \) and \( t = 1, 2 \), by (3.8),
\[ \int_{\Omega_p} \chi_i Z_{i0} \tilde{Z}_{i0} = \int_{\mathbb{R}^2} \chi Z_0^2 = C_0 > 0, \quad \int_{\Omega_p} \chi_i^2 Z_{i0}^2 = \int_{\mathbb{R}^2} \chi^2 Z_0^2 = 0, \]
while if \( i = l + 1, \ldots, m \) and \( t = 1 \), by (3.10), (3.12) and (3.14),
\[
\int_{\Omega_p} \chi_i Z_{i0} \tilde{Z}_{i0} = \int_{\mathbb{R}^n_{+}} \chi_i Z_{i0}^2 [1 + O(\varepsilon \mu_i |z|)] = \frac{C_0}{2} + O(\varepsilon \mu_i),
\]
\[
\int_{\Omega_p} \chi_i^2 Z_{i0} Z_{i0} = \int_{\mathbb{R}^n_{+}} \chi_i^2 Z_{i0}^2 [1 + O(\varepsilon \mu_i |z|)] = O(\varepsilon \mu_i).
\]

Let us denote \( \mathcal{A} \) the coefficient matrix of systems (3.39)-(3.40) with respect to \( (d_q, d_1, \ldots, d_m) \). From the above estimates it follows that \( P^{-1} \mathcal{A} P \) is diagonally dominant and then invertible, where \( P = \text{diag}(\rho_0 v_0 / \varepsilon, \mu_1, \ldots, \mu_m) \). Hence \( \mathcal{A} \) is also invertible and \( (d_q, d_1, \ldots, d_m) \) is well defined.

Estimate (3.27) is a direct consequence of the following two claims.

**Claim 1.** Let \( \mathcal{L} = -\Delta_{a(\varepsilon y)} + \varepsilon^2 - W_{\varepsilon'} \), then for any \( i = 1, \ldots, m \) and \( j = 1, J_i \),
\[
\| \mathcal{L}(\tilde{Z}_i) \|_* \leq \frac{C \varepsilon \log p}{p \rho_0 v_0},
\]
and
\[
\| \mathcal{L}(\chi_i Z_{ij}) \|_* \leq \frac{C}{\mu_i}, \quad \| \mathcal{L}(\tilde{Z}_{i0}) \|_* \leq \frac{C \log p}{p \mu_i}.
\]

**Claim 2.** For any \( i = 1, \ldots, m \) and \( j = 1, J_i \),
\[
|d_q| \leq C \frac{\rho_0 v_0}{\varepsilon} \| h \|_*, \quad |d_i| \leq C \rho_i \| h \|_*, \quad |e_{ij}| \leq C \mu_i \log p \| h \|_*.
\]

In fact, by the definition of \( \tilde{\phi} \) in (3.36) we obtain
\[
\begin{cases}
\mathcal{L}(\tilde{\phi}) = h + d_q \mathcal{L}(\tilde{Z}_q) + \sum_{i=1}^m d_i \mathcal{L}(\tilde{Z}_{i0}) + \sum_{i=1}^m \sum_{j=1}^{J_i} e_{ij} \mathcal{L}(\chi_i Z_{ij}) & \text{in } \Omega_p, \\
\frac{\partial \tilde{\phi}}{\partial \nu} = 0 & \text{on } \partial \Omega_p.
\end{cases}
\]

Since the orthogonality conditions in (3.17) hold, by estimate (3.18), Claims 1 and 2 we conclude
\[
\| \tilde{\phi} \|_{L^\infty(\Omega_p)} \leq C \left\{ \| h \|_* + |d_q| \| \mathcal{L}(\tilde{Z}_q) \|_* + \sum_{i=1}^m |d_i| \| \mathcal{L}(\tilde{Z}_{i0}) \|_* + \sum_{i=1}^m \sum_{j=1}^{J_i} |e_{ij}| \| \mathcal{L}(\chi_i Z_{ij}) \|_* \right\} \leq C \log p \| h \|_*.
\]

Using the definition of \( \tilde{\phi} \) again and the fact that
\[
\| \tilde{Z}_q \|_{L^\infty(\Omega_p)} \leq \frac{C \varepsilon}{\rho_0 v_0}, \quad \| \tilde{Z}_{i0} \|_{L^\infty(\Omega_p)} \leq \frac{C}{\mu_i}, \quad \| \chi_i Z_{ij} \|_{L^\infty(\Omega_p)} \leq \frac{C}{\mu_i}, \quad i = 1, \ldots, m, \quad j = 1, J_i,
\]
estimate (3.27) then follows from estimate (3.45) and Claim 2.

**Proof of Claim 1.** Let us try with inequality (3.41). Fixing \( q \in \partial \Omega \) and observing that \( F^p_q(q') = (0, 0) \) and \( \nabla F^p_q(q') = A_q \), by (3.9) and (3.11) we find
\[
z_q := F^p_q(y) = \frac{1}{\varepsilon} F_q(A_q (\varepsilon y - q)) = A_q(y-q') \left[ 1 + O(\varepsilon A_q(y-q')) \right],
\]
and
\[
\nabla y = A_q \nabla z_q + O(\varepsilon |z_q|) \nabla z_q \quad \text{and} \quad -\Delta y = -\Delta z_q + O(\varepsilon) \nabla^2 z_q + O(\varepsilon) \nabla z_q.
\]
Furthermore,
\[
-\Delta a(y) = -\frac{1}{a(\varepsilon y)} \nabla y (a(\varepsilon y) \nabla y (\cdot)) = -\Delta y - \varepsilon \nabla \log a(\varepsilon y) \nabla y = -\Delta z_q + O(\varepsilon |z_q|) \nabla^2 z_q + O(\varepsilon) \nabla z_q.
\]
Thus by (3.2) and (3.13),
\[ Z_q - \frac{\varepsilon}{\rho_0 v_0} = - \frac{2\varepsilon}{\rho_0 v_0} \left[ 1 + \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{2(1+\alpha)} = O \left( \frac{\varepsilon}{\rho_0 v_0} \left[ 1 + \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{-2(1+\alpha)} \right), \]  
(3.50)
and
\[ \Delta_{a(y)} Z_q + \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1+\alpha)^2 |\varepsilon y-q|^2}{1 + \frac{|\varepsilon y-q|}{\rho_0 v_0}^{2(1+\alpha)}} Z_q = O \left( \frac{\varepsilon}{\rho_0 v_0} \left[ \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{2(1+\alpha)} \right), \]  
(3.51)
Consider the four regions
\[ \Omega_1 = (F^q_p)^{-1} \left( \left\{ \frac{\varepsilon y}{\rho_0 v_0} \leq R \right\} \cap \mathbb{R}_+^2 \right), \quad \Omega_2 = (F^q_p)^{-1} \left( \left\{ R < \frac{\varepsilon y}{\rho_0 v_0} \leq R + 1 \right\} \cap \mathbb{R}_+^2 \right), \]
\[ \Omega_3 = (F^q_p)^{-1} \left( \left\{ R + 1 < \frac{|\varepsilon y-q|}{\rho_0 v_0} \leq \frac{3d}{\rho_0 v_0} \right\} \cap \mathbb{R}_+^2 \right), \quad \Omega_4 = (F^p_q)^{-1} \left( \left\{ \frac{3d}{\rho_0 v_0} < \frac{|\varepsilon y-q|}{\rho_0 v_0} \leq \frac{6d}{\rho_0 v_0} \right\} \cap \mathbb{R}_+^2 \right). \]
Notice that for any \( y \in \Omega_1 \cup \Omega_2 \), by (2.43) and (3.13),
\[ \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1+\alpha)^2 |\varepsilon y-q|^2}{1 + \frac{|\varepsilon y-q|}{\rho_0 v_0}^{2(1+\alpha)}} Z_q = O \left( \frac{\varepsilon}{\rho_0 v_0} \left[ \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{2(1+\alpha)} \right), \]  
(3.52)
and for any \( y \in \Omega_2 \cup \Omega_3 \cup \Omega_4 \) and any \( \beta \in (0, 1) \), by (1.9), (2.20), (3.22) and (3.29),
\[ Z_q - \tilde{Z}_q = \frac{\varepsilon}{\rho_0 v_0} - a_q G(\varepsilon, y, q) = a_q \left[ \frac{4(1+\alpha)}{c_0} \log \left( \frac{|\varepsilon y-q|}{R \rho_0 v_0} \right) + O \left( \frac{|\varepsilon y-q|^\beta}{R \rho_0 v_0} \right) \right]. \]  
(3.53)
In \( \Omega_1 \), by (3.51) and (3.52),
\[ \mathcal{L}(\tilde{Z}_q) = \mathcal{L}(Z_q) = O \left( \frac{1}{p} \left( \frac{\varepsilon}{\rho_0 v_0} \right)^3 \left[ \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{2\alpha} \right). \]  
(3.54)
In \( \Omega_2 \), by (1.8), (3.28) and (3.35),
\[ \mathcal{L}(\tilde{Z}_q) = \mathcal{L}(Z_q) - (1 - \eta_1) \mathcal{L}(Z_q - \tilde{Z}_q) - 2 \nabla \eta_1 \nabla (Z_q - \tilde{Z}_q) - (Z_q - \tilde{Z}_q) \Delta_{a(y)} \eta_1 \]
\[ = \left[ -\Delta_{a(y)} Z_q - \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1+\alpha)^2 |\varepsilon y-q|^2}{1 + \frac{|\varepsilon y-q|}{\rho_0 v_0}^{2(1+\alpha)}} Z_q \right] + \left[ \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1+\alpha)^2 |\varepsilon y-q|^2}{1 + \frac{|\varepsilon y-q|}{\rho_0 v_0}^{2(1+\alpha)}} \right] - W_c \right] Z_q \]
\[ + \varepsilon^2 \left( Z_q - \frac{\varepsilon}{\rho_0 v_0} \right) + \frac{\varepsilon^3}{\rho_0 v_0} \eta_1 + W_c (1 - \eta_1) (Z_q - \tilde{Z}_q) - 2 \nabla \eta_1 \nabla (Z_q - \tilde{Z}_q) - (Z_q - \tilde{Z}_q) \Delta_{a(y)} \eta_1. \]
Notice that in \( \Omega_2 \), by (3.29), (3.30), (3.47) and (3.53),
\[ |Z_q - \tilde{Z}_q| = O \left( \frac{\varepsilon^2}{p \rho_0 v_0 R} \right) \quad \text{and} \quad \left| \nabla (Z_q - \tilde{Z}_q) \right| = O \left( \frac{\varepsilon^2}{p \rho_0 v_0^2 R} \right). \]  
(3.55)
Moreover, \(|\nabla \eta_1| = O(\varepsilon/\rho_0 v_0)| and |\Delta_{a(y)} \eta_1| = O(\varepsilon^2/\rho_0 v_0^2)|. From (2.43), (3.50)-(3.52) and (3.55) we obtain
\[ \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon^3}{p \rho_0 v_0^2 R} \right). \]  
(3.56)
In \( \Omega_3 \), by (1.8), (3.28), (3.35), (3.50) and (3.51),
\[ \mathcal{L}(\tilde{Z}_q) = \mathcal{L}(Z_q) - \mathcal{L}(Z_q - \tilde{Z}_q) \]
\[ \equiv A_1 + A_2 + O \left( \frac{\varepsilon}{\rho_0 v_0} \left[ 1 + \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{-3-2\alpha} \right) + O \left( \frac{\varepsilon^3}{\rho_0 v_0} \left[ 1 + \frac{|\varepsilon y-q|}{\rho_0 v_0} \right]^{-2(1+\alpha)} \right), \]
where

\[
A_1 = \left[ \frac{\varepsilon}{(\rho_0 v_0)^2} \right] 8(1 + \alpha)^2 \frac{|y - q|^{2\alpha}}{\rho_0 v_0 - |W_\delta|^2} - W_\delta \right] Z_q \quad \text{and} \quad A_2 = W_\delta \left[ \frac{\varepsilon}{\rho_0 v_0} - a_q G(z, y, q) \right].
\]

For the estimates of these two terms, we set \( z_k := y - \xi_k \) for all \( k = 1, \ldots, l \), but \( z_k := F_k^p(y) \) for all \( k = l+1, \ldots, m \), and divide \( \Omega_3 \) into several pieces:

\[
\Omega_{3, k} = \Omega_3 \cap \left\{ \frac{z_k}{\mu_k} \leq \frac{1}{3 \rho_0 v_0} \right\} \quad \forall \ k = 1, \ldots, l,
\]

\[
\Omega_{3, k} = \Omega_3 \cap \left( F_k^p \right)^{-1} \left\{ \left\{ \frac{z_k}{\mu_k} \leq \frac{1}{3 \rho_0 v_0} \right\} \cap \mathbb{R}^2_+ \right\} \quad \forall \ k = l + 1, \ldots, m,
\]

and

\[
\Omega_q = \left( F_k^p \right)^{-1} \left\{ \frac{R + 1}{3 \rho_0 v_0} \leq \frac{1}{3 \rho_0 v_0} \right\} \cap \mathbb{R}^2_+ \quad \text{and} \quad \tilde{\Omega}_3 = \Omega_3 \setminus \bigcup_{k=1}^m \Omega_{3, k} \cup \Omega_q.
\]

Note that for any \( k = l + 1, \ldots, m \), by (3.10) and (3.12),

\[
z_k = F_k^p(y) = \frac{1}{\varepsilon} F_k(A_k(z, y - \xi_k)) = A_k(z, y - \xi_k) \left[ 1 + O(E_k(z, y - \xi_k)) \right].
\]

From (2.1), (2.12), (2.41), (2.43), (3.47) and (3.57) we have

\[
A_1 = \left\{ \left. \frac{\varepsilon}{(\rho_0 v_0)^2} \left| \frac{y - q}{\rho_0 v_0} \right|^{-4 - 2\alpha} \mathcal{O} \left( \frac{1}{p} \log \frac{\left| \frac{y - q}{\rho_0 v_0} \right|}{R \rho_0 v_0} \right) \right| \right\} \quad \text{in} \ \Omega_q,
\]

and

\[
A_2 = \left( \frac{\varepsilon}{(\rho_0 v_0)^3} \left| \frac{y - q}{\rho_0 v_0} \right|^{-4 - 2\alpha} \mathcal{O} \left( \frac{1}{p} \log \frac{\left| \frac{y - q}{\rho_0 v_0} \right|}{R \rho_0 v_0} \right) \right) \quad \text{in} \ \tilde{\Omega}_3.
\]

Thus in \( \Omega_q \cup \tilde{\Omega}_3 \),

\[
\mathcal{L} (\tilde{Z}_q) = \mathcal{L} (\tilde{Z}_q) = \left( \frac{\varepsilon}{(\rho_0 v_0)^2} \left| \frac{y - q}{\rho_0 v_0} \right|^{-4 - 2\alpha} \mathcal{O} \left( \frac{1}{p} \log \frac{\left| \frac{y - q}{\rho_0 v_0} \right|}{R \rho_0 v_0} \right) \right) \quad \text{in} \ \Omega_q \cup \tilde{\Omega}_3.
\]

In \( \Omega_{3, k} \) with \( k = 1, \ldots, m \), by (2.42), (3.31), (3.50), (3.51) and (3.57),

\[
\mathcal{L} (\tilde{Z}_q) = \mathcal{L} (\tilde{Z}_q) = \left( \frac{\varepsilon}{(\rho_0 v_0)^2} \left| \frac{y - q}{\rho_0 v_0} \right|^{-4 - 2\alpha} \mathcal{O} \left( \frac{1}{p} \log \frac{\left| \frac{y - q}{\rho_0 v_0} \right|}{R \rho_0 v_0} \right) \right) \quad \text{in} \ \Omega_q \cup \tilde{\Omega}_3.
\]

In \( \Omega_q \), by (1.8), (3.28) and (3.35),

\[
\mathcal{L} (\tilde{Z}_q) = \left( \frac{\varepsilon}{(\rho_0 v_0)^2} \left| \frac{y - q}{\rho_0 v_0} \right|^{-4 - 2\alpha} \mathcal{O} \left( \frac{1}{p} \log \frac{\left| \frac{y - q}{\rho_0 v_0} \right|}{R \rho_0 v_0} \right) \right) \quad \text{in} \ \Omega_q \cup \tilde{\Omega}_3.
\]
From the previous choice of the number \( d \) we have that for any \( y \in \Omega_4 \) and any \( k = 1, \ldots, m \), by (2.5) and (3.47),
\[
|y - \xi_k| \geq |y - q'| - |q' - \xi_k| \geq \frac{2d}{\varepsilon} - \frac{d}{\varepsilon} = \frac{\sqrt{d_i}}{\varepsilon p^{2\kappa}}.
\]
Then by (2.28) and (2.41) we find \( W_{c'} = O(\varepsilon^{4-\beta}) \) in \( \Omega_4 \). In addition, \(|\nabla \eta_{q2}| = O(\varepsilon/d)\), \(|\Delta_{a(\varepsilon y)} \eta_{q2}| = O(\varepsilon^2/d^2)\),
\[
|\tilde{Z}_q| = O\left(\frac{\varepsilon |\log d|}{p \rho_0 v_0}\right) \quad \text{and} \quad |\nabla \tilde{Z}_q| = O\left(\frac{\varepsilon^2}{pd \rho_0 v_0}\right) \quad \text{in} \quad \Omega_4.
\]
Furthermore, by (3.50) and (3.51),
\[
\mathcal{L}(\tilde{Z}_q) = O\left(\frac{\varepsilon^3 |\log d|}{pd^2 \rho_0 v_0}\right).
\]
Hence by (3.54), (3.56), (3.58), (3.59), (3.61) and the definition of \( \|\cdot\|_* \) with respect to (2.39), we arrive at
\[
\|\mathcal{L}(\tilde{Z}_q)\|_* = O\left(\frac{\varepsilon \log p}{p \rho_0 v_0}\right).
\]

The inequalities in \(3.42\) are easy to establish as they are very similar to the consideration of inequality \(3.41\), so we leave the detailed proof to readers.

**Proof of Claim 2.** Testing equation (3.44) against \( a(\varepsilon y)\tilde{Z}_q \) and using estimates (3.45)-(3.46), we can derive that
\[
d_q \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_q \mathcal{L}(\tilde{Z}_q) + \sum_{k=1}^m d_k \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_{k0} \mathcal{L}(\tilde{Z}_q)
\]
\[
= - \int_{\Omega_p} a(\varepsilon y)h\tilde{Z}_q + \int_{\Omega_p} a(\varepsilon y)\bar{\phi} \mathcal{L}(\tilde{Z}_q) - \sum_{k=1}^m \sum_{i=1}^{J_k} e_{ki} \int_{\Omega_p} a(\varepsilon y)\chi_i \tilde{Z}_{k}\mathcal{L}(\tilde{Z}_q)
\]
\[
\leq \frac{C \varepsilon}{\rho_0 v_0} \|h\|_* + C \|\mathcal{L}(\tilde{Z}_q)\|_* \left(\|\bar{\phi}\|_{L^\infty(\Omega_p)} + \sum_{k=1}^m \sum_{i=1}^{J_k} \frac{1}{\mu_k} |e_{ki}|\right)
\]
\[
\leq \frac{C \varepsilon}{\rho_0 v_0} \|h\|_* + C \|\mathcal{L}(\tilde{Z}_q)\|_* \left[ \|h\|_* + \|d_q\| \|\mathcal{L}(\tilde{Z}_q)\|_* + \sum_{k=1}^m \|d_k\| \|\mathcal{L}(\tilde{Z}_{k0})\|_* + \sum_{k=1}^m \sum_{i=1}^{J_k} \frac{1}{\mu_k} \left( |e_{ki}| + \|\mathcal{L}(\chi_i \tilde{Z}_{k0})\|_* \right)\right],
\]
where we have applied the following two inequalities:
\[
\left(\frac{\varepsilon}{p \rho_0 v_0}\right)^2 \int_{\Omega_p} \frac{\varepsilon \mu^2}{\rho_0 v_0^3} \frac{dy}{\left(1 + \frac{y_0}{\rho_0 v_0}\right)^{4+2\alpha}} \leq C \quad \text{and} \quad \int_{\Omega_p} \frac{1}{\mu^2} \frac{dy}{\left(1 + \frac{y_0}{\rho_0 v_0}\right)^{4+2\alpha}} \leq C, \quad i = 1, \ldots, m.
\]
This, combined with inequality \(3.38\) and Claim 1, gives
\[
|d_q| \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_q \mathcal{L}(\tilde{Z}_q) \leq \frac{C \varepsilon}{\rho_0 v_0} \|h\|_* + \frac{C \varepsilon \log^2 p}{p \rho_0 v_0} \left( \frac{1}{\mu} \sum_{k=1}^m \|d_k\| \right) + \sum_{k=1}^m \|d_k\| \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_{k0} \mathcal{L}(\tilde{Z}_q) \|\| \|. \quad \text{(3.62)}
\]
Similarly, testing (3.44) against \( a(\varepsilon y)\tilde{Z}_i \), \( i = 1, \ldots, m \) and using \(3.38\), \(3.45\), \(3.46\) and Claim 1, we find
\[
|d_i| \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_i \mathcal{L}(\tilde{Z}_i) \leq \frac{C \varepsilon}{\mu_i} \|h\|_* + \frac{C \varepsilon \log^2 p}{p \mu_i} \left( \frac{1}{\mu} \sum_{k=1}^m \|d_k\| \right) + \|d_q\| \int_{\Omega_p} a(\varepsilon y)\tilde{Z}_{i0} \mathcal{L}(\tilde{Z}_i) \|\| \|. \quad \text{(3.63)}
\]
To achieve the estimates of \(d_q, d_i\) and \( e_{ij} \) in \(3.43\), we have the following claim.
Claim 3. If $d$ is sufficiently small, but $R$ is sufficiently large, then we have that for any $i, k = 1, \ldots, m$ with $i \neq k$,
\[
\int_{\Omega_p} a(\xi) \mathcal{L}(\tilde{z}_i) = \frac{2c_i a(\xi)}{p \mu_i} \left[ 1 + O \left( \frac{1}{R^2} \right) \right],
\]
and
\[
\int_{\Omega_p} a(\xi) \mathcal{L}(\tilde{z}_k) = O \left( \frac{\log^2 p}{p^2 \mu_i \mu_k} \right),
\]
where the coefficients $c_0$ and $c_i, i = 1, \ldots, m$ are defined in (2.20).

In fact, once Claim 3 is valid, then inserting (3.64) and (3.65) into (3.63) and (3.62), respectively, we give
\[
\frac{\varepsilon |d_k|}{p \rho_0 v_0} \leq C h \|s\| + \frac{C \log^2 p}{p} \left( \frac{\varepsilon |d_k|}{p \rho_0 v_0} + \sum_{k=1}^{m} \frac{|d_k|}{p \mu_k} \right),
\]
and for any $i = 1, \ldots, m$,
\[
\frac{|d_i|}{p \mu_i} \leq C h \|s\| + \frac{C \log^2 p}{p} \left( \frac{\varepsilon |d_k|}{p \rho_0 v_0} + \sum_{k=1}^{m} \frac{|d_k|}{p \mu_k} \right).
\]
As a result, using linear algebra arguments for (3.66)-(3.67), we can prove Claim 2 for $d_k$ and $d_i$, and then complete the proof by inequality (3.38).

Proof of Claim 3. Let us first establish the validity of the two expansions in (3.65). Observe that
\[
\mathcal{L}(\tilde{z}_i) = \eta_1 \mathcal{L}(Z_i - \tilde{z}_i) + \eta_2 \mathcal{L}(\tilde{z}_i) = (Z_i - \tilde{z}_i) \Delta_{a(\xi)} \eta_1 - 2 \nabla \eta_1 \nabla (Z_i - \tilde{z}_i) - 2 \nabla \eta_2 \nabla \tilde{z}_i \Delta_{a(\xi)} \eta_2.
\]
Thus by (3.28) and (3.35),
\[
\int_{\Omega_p} a(\xi) \mathcal{L}(\tilde{z}_i) = K + L,
\]
where
\[
K = \int_{\Omega_p} a(\xi) \mathcal{L}(Z_i - \tilde{z}_i) + \eta_2 \mathcal{L}(\tilde{z}_i)
\]
and
\[
L = \int_{\Omega_p} a(\xi) \mathcal{L}(Z_i - \tilde{z}_i) \left[ 1 - \eta_1 \mathcal{L}(Z_i - \tilde{z}_i) + \eta_2 \mathcal{L}(\tilde{z}_i) \right]
\]
Integrating by parts the first term and the last term of $K$, respectively, we obtain
\[
K = - \int_{\Omega_2} a(\xi) \nabla \eta_1 \nabla (Z_i - \tilde{z}_i) + \int_{\Omega_2} a(\xi) (Z_i - \tilde{z}_i) \nabla \eta_1 \nabla (Z_i - \tilde{z}_i)
\]
\[
+ \int_{\Omega_2} a(\xi) (Z_i - \tilde{z}_i) \nabla \eta_1 |^2 + \int_{\Omega_2} a(\xi) (Z_i - \tilde{z}_i) \nabla \eta_2 |^2 + \int_{\Omega_4} a(\xi) |\tilde{z}_i|^2 |\nabla \eta_2|^2
\]
\[
\equiv K_{21} + K_{22} + K_{23} + K_{24} + K_4.
\]
From (2.5), (2.20), (2.28), (3.2), (3.13), (3.28), (3.29), (3.32), (3.47), (3.48) and (3.53) we can compute

\[ K_{21} = -a \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \int_{(R^+)^{-1} \left( \left\{ R < \left| \frac{\rho_0 v_0}{\rho_0 v_0} \right| \leq R+1 \right\} \cap \mathbb{R}^2_+ \right)} \frac{1}{y-q} a(\varepsilon y) \frac{\varepsilon y_1}{\rho_0 v_0} \eta_1 \left( \frac{\varepsilon y_1}{\rho_0 v_0} \right) \left[ \frac{4(1+\alpha)}{c_0} + o(1) \right] dy \]

\[ = -\frac{c_0 a(q)}{4(1+\alpha) \rho_0 v_0} \int_R^{R+1} a(q) \eta_1'(r) \left[ \frac{4(1+\alpha)}{c_0} + O \left( \frac{1}{r^{2(1+\alpha)}} \right) \right] dr \]

\[ = \frac{c_0 a(q)}{p} \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \left[ 1 + O \left( \frac{1}{R^{2(1+\alpha)}} \right) \right]. \]

Moreover by (3.13), (3.32), (3.47) and (3.53), we find \( |\nabla \eta_1| = O \left( \frac{\varepsilon}{\rho_0 v_0} \right) \) and \( |\nabla \hat{Z}_\varepsilon| = O \left( \frac{\varepsilon^2}{\rho_0 v_0^2 R^{3+2\alpha}} \right) \) in \(\Omega_2\). Furthermore, by (3.55),

\[ K_{22} = O \left( \frac{\varepsilon^2}{p^2 \rho_0^2 v_0^2 R} \right), \quad K_{23} = O \left( \frac{\varepsilon^2}{p^2 \rho_0^2 v_0^2 R} \right), \quad K_{24} = O \left( \frac{\varepsilon^2}{p^2 \rho_0^2 v_0^2 R^{3+2\alpha}} \right). \]

By (3.60),

\[ K_4 = O \left( \frac{\varepsilon^2 |\log d|^2}{p^2 \rho_0^2 v_0^2} \right). \]

Hence for \( R \) and \( p \) large enough, but \( d \) small enough,

\[ K = \frac{c_0 a(q)}{p} \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \left[ 1 + O \left( \frac{1}{R^{2(1+\alpha)}} \right) \right]. \]  

(3.69)

As for the asymptotic behavior of \( L \), by virtue of (3.2), (3.13), (3.28)-(3.32), (3.47) and (3.53) we have, by (2.43),

\[ \int_{(F^\varepsilon)} a(\varepsilon y) \eta_2^2 \left[ Z_q - (1-\eta_1) \left( \frac{\varepsilon}{\rho_0 v_0} - a \eta_1 G(\varepsilon y, q) \right) \right] \times (1-\eta_1) W_\varepsilon' \left( \frac{\varepsilon}{\rho_0 v_0} - a \eta_1 G(\varepsilon y, q) \right) dy \]

\[ = O \left( \frac{\varepsilon^2}{p \rho_0^2 v_0^2 R^{2(1+\alpha)}} \right), \]

and by (3.50),

\[ \int_{(F^\varepsilon)} a(\varepsilon y) \eta_2^2 \left[ Z_q - (1-\eta_1) \left( \frac{\varepsilon}{\rho_0 v_0} - a \eta_1 G(\varepsilon y, q) \right) \right] \times \left[ \varepsilon^2 \left( Z_q - \frac{\varepsilon}{\rho_0 v_0} \right) + \frac{\varepsilon^3}{\rho_0 v_0} \eta_1 \right] dy \]

\[ = O \left( p \varepsilon^2 \right). \]
and by (3.51),
\[
\int (p^q)^{-1} \left( \left\{ \frac{\varepsilon_y}{\rho_0 v_0} \leq \frac{1}{3 \rho_0^2 |y|} \right\} \cap \mathbb{R}^2_+ \right) a(\varepsilon y) \eta_{q2}^2 \left[ Z_q - (1 - \eta_{q1}) \left( \frac{\varepsilon}{\rho_0 v_0} - a_y G(\varepsilon y, q) \right) \right] dy
\times \left[ \Delta_{\alpha(\varepsilon y)} Z_q + \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1 + \alpha)^2 |\varepsilon y - q|^{2\alpha}}{(1 + |\varepsilon y - q|)^{2(1 + \alpha)}} Z_q \right] dy
= O \left( \frac{\varepsilon^2}{\rho_0 v_0} \right),
\]

and by (2.43),
\[
\int (p^q)^{-1} \left( \left\{ \frac{\varepsilon_y}{\rho_0 v_0} \leq \frac{1}{3 \rho_0^2 |y|} \right\} \cap \mathbb{R}^2_+ \right) a(\varepsilon y) \eta_{q2}^2 (1 - \eta_{q1}) \left( \frac{\varepsilon}{\rho_0 v_0} - a_y G(\varepsilon y, q) \right) \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \frac{8(1 + \alpha)^2 |\varepsilon y - q|^{2\alpha}}{(1 + |\varepsilon y - q|)^{2(1 + \alpha)}} - W_{\varepsilon} \right] Z_q dy
= O \left( \frac{\varepsilon^2}{p^2 \rho_0^2 v_0^2 R^{1 + \alpha}} \right),
\]

while by (3.31), (3.58) and (3.59),
\[
\int (p^q)^{-1} \left( \left\{ \frac{\varepsilon_y}{\rho_0 v_0} \leq \frac{1}{3 \rho_0^2 |y|} \right\} \cap \mathbb{R}^2_+ \right) a(\varepsilon y) \tilde{Z}_q \left[ \eta_{q1} L(Z_q - \tilde{Z}_q) + \eta_{q2} L(\tilde{Z}_q) \right] dy
= \sum_{k=1}^{m} \int_{\Omega_{\varepsilon}(k)} a(\varepsilon y) \tilde{Z}_q \left[ \eta_{q1} L(Z_q - \tilde{Z}_q) + \eta_{q2} L(\tilde{Z}_q) \right] dy
= \sum_{k=1}^{m} O \left( \int_{\mu_k/(p^{2\alpha} \sqrt{\varepsilon y})} \frac{1}{P_k} \frac{8 \varepsilon^2 \log^2 p}{(1 + |\varepsilon y - q|)^{2(1 + \alpha)}} - W_{\varepsilon} \right) dy
+ O \left( \int_{\mu_k/(p^{2\alpha} \sqrt{\varepsilon y})} \frac{4(1 + \alpha)^2 |\varepsilon y - q|}{\rho_0 v_0} \log \frac{|\varepsilon y - q|}{\rho_0 v_0} + O \left( |\varepsilon y - q|^2 \right) O(1) \right) dy
= O \left( \frac{\varepsilon^2 \log^2 p}{p^2 \rho_0^2 v_0^2} \right).
\]

Then
\[
L = \int (p^q)^{-1} \left( \left\{ \frac{\varepsilon_y}{\rho_0 v_0} \leq \frac{1}{3 \rho_0^2 |y|} \right\} \cap \mathbb{R}^2_+ \right) a(\varepsilon y) \eta_{q2}^2 Z_q^2 \left[ \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 8(1 + \alpha)^2 |\varepsilon y - q|^{2\alpha} \right] \frac{1}{(1 + |\varepsilon y - q|)^{2(1 + \alpha)}} - W_{\varepsilon} \right] dy + O \left( \frac{\varepsilon^2}{p \rho_0^2 v_0^2 R^{2(1 + \alpha)}} \right).
In a straightforward but tedious way, by (2.1), (2.16) and (3.2) we can compute
\[
\int_{\mathbb{R}^2} z^{2\alpha} e^{U_1} |\nabla z|^2 \left( \omega_1 - U_1 - \frac{1}{2} U_1^2 \right) dz = -4\pi (1 + \alpha).
\]
Thus by (2.20), (2.43), (3.13), (3.47) and (3.69), we conclude that for \( R \) and \( p \) large enough, but \( d \) small enough,
\[
\int_{\Omega_{\varepsilon}} a(\varepsilon y) \tilde{Z}_q \mathcal{L}(\tilde{Z}_q) = K + L = \frac{2c_0 a(q)}{p} \left( \frac{\varepsilon}{\rho_0 v_0} \right)^2 \left[ 1 + O \left( \frac{1}{R^{21+\alpha} \log R} \right) \right].
\]
(3.70)
Let us calculate \( \int_{\Omega_{\varepsilon}} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) \) for all \( k = 1, \ldots, m \). From the previous estimates of \( \mathcal{L}(\tilde{Z}_q) \) and \( \tilde{Z}_{k_0} \), we can easily prove that
\[
\int_{\Omega_1} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon R^{2(1+\alpha)} \log p}{p^2 \rho_0 v_0 \mu_k} \right), \quad \int_{\Omega_2} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon \log p}{p^2 \rho_0 v_0 \mu_k} \right),
\]
\[
\int_{\Omega_3} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon \log d^2}{p^2 \rho_0 v_0 \mu_k} \right), \quad \int_{\Omega_{\varepsilon} \setminus \Omega_3} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon \log p}{p^2 \rho_0 v_0 \mu_k} \right),
\]
and
\[
\int_{\Omega_{\varepsilon},l} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = O \left( \frac{\varepsilon \log^2 p}{p^2 \rho_0 v_0 \mu_k} \right) \quad \text{for all } l \neq k.
\]
It remains to consider the integral over \( \Omega_{3,k} \). Using (3.35) and an integration by parts, we obtain
\[
\int_{\Omega_{3,k}} a(\varepsilon y) \tilde{Z}_{k_0} \mathcal{L}(\tilde{Z}_q) = \int_{\Omega_{3,k}} a(\varepsilon y) \tilde{Z}_q \mathcal{L}(\tilde{Z}_{k_0}) - \int_{\partial \Omega_{3,k}} a(\varepsilon y) \tilde{Z}_{k_0} \frac{\partial \tilde{Z}_q}{\partial \nu} + \int_{\partial \Omega_{3,k}} a(\varepsilon y) \tilde{Z}_q \frac{\partial \tilde{Z}_{k_0}}{\partial \nu}.
\]
Notice that
\[
\int_{\Omega_{3,k}} a(\varepsilon y) \tilde{Z}_q \mathcal{L}(\tilde{Z}_{k_0}) = \left( \int_{\left\{ \frac{|z_k|}{\mu_k} \leq R \right\}} + \int_{\left\{ R < |z_k| \leq R+1 \right\}} + \int_{\left\{ R+1 < |z_k| \leq \frac{R + 1}{\mu_k} \right\}} a(\varepsilon y) \tilde{Z}_q \mathcal{L}(\tilde{Z}_{k_0}) \right).
\]
By (2.42), (3.2), (3.8), (3.14), (3.28)-(3.31), (3.35) and (3.57) we can compute that for any \( |z_k| \leq \mu_k R \),
\[
\mathcal{L}(\tilde{Z}_{k_0}) = \mathcal{L}(Z_{k_0}) = O \left( \frac{1}{p \mu_k^3} \right),
\]
for any \( \mu_k R < |z_k| \leq \mu_k (R + 1) \),
\[
\mathcal{L}(\tilde{Z}_{k_0}) = O \left( \frac{1}{p \mu_k^3 R} \right),
\]
and for any \( \mu_k (R + 1) < |z_k| \leq \mu_k (3p^2 \sqrt{\varepsilon} \mu_k) \),
\[
\mathcal{L}(\tilde{Z}_{k_0}) = \mathcal{L}(\tilde{Z}_{k_0}) = \frac{1}{\mu_k^2} \left| \frac{y - \xi_k}{\mu_k} \right|^{-4} O \left( \frac{1}{p \log \left| \frac{y - \xi_k}{R \mu_k} \right|} \right).
\]
These, combined with the estimate of \( \tilde{Z}_q \) in (3.31), imply
\[
\int_{\Omega_{3,k}} a(\varepsilon y) \tilde{Z}_q \mathcal{L}(\tilde{Z}_{k_0}) = O \left( \frac{\varepsilon \log p}{p^2 \rho_0 v_0 \mu_k} \right).
\]
As on \( \partial \Omega_{3,k} \), by (2.3) and (3.31),
\[
\tilde{Z}_{k_0} = O \left( \frac{1}{\mu_k} \right), \quad \tilde{Z}_q = O \left( \frac{\varepsilon \log p}{p \rho_0 v_0} \right),
\]
and
\[
|\nabla \tilde{Z}_{k_0}| = O \left( \frac{\varepsilon^{1/2} \mu_k^{-1}}{\mu_k^{1/2}} \right), \quad |\nabla \tilde{Z}_q| = O \left( \frac{\varepsilon^2 \mu_k^{-1}}{\rho_0 v_0} \right).
\]
Then
\[ \int_{\Omega_{\alpha}} a(\varepsilon y) \bar{Z}_{k0} \mathcal{L}(\bar{Z}_q) = O \left( \frac{\varepsilon \log p}{p^2 \rho_0 v_0 \mu_k} \right). \]

By the above estimates, we readily have
\[ \int_{\Omega_p} a(\varepsilon y) \bar{Z}_{k0} \mathcal{L}(\bar{Z}_q) = O \left( \frac{\varepsilon \log^2 p}{p^2 \rho_0 v_0 \mu_k} \right), \quad k = 1, \ldots, m. \tag{3.71} \]

The two expansions in (3.64) are easy to establish as they are very similar to the above consideration for the two expansions in (3.65), so we leave the detailed proof to readers. \( \square \)

**Step 4:** Proof of Proposition 3.1. We try with establishing the validity of the a priori estimate
\[ \| \phi \|_{L^\infty(\Omega_p)} \leq C p \| h \|_* \]  
for any \( \phi, c_{ij} \) solutions of problem (3.1) and any \( h \in C^{0,\alpha}(\Omega_p) \). The previous step gives
\[ \| \phi \|_{L^\infty(\Omega_p)} \leq C p \left( \| h \|_* + \sum_{i=1}^m \sum_{j=1}^{J_i} |c_{ij}| \cdot \| \chi_i Z_{ij} \|_* \right) \leq C p \left( \| h \|_* + \sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| \right). \]

As before, arguing by contradiction to (3.72), we can proceed as in Step 2 and suppose further that
\[ \| \phi_n \|_{L^\infty(\Omega_{pn})} = 1, \quad p_n \| h_n \|_* \to 0, \quad p_n \sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| \geq d > 0 \quad \text{as } n \to +\infty. \tag{3.73} \]

For simplicity of argument we omit the dependence on \( n \) and consider the case \( q \in \partial \Omega \) only. It suffices to estimate the values of the constants \( c_{ij} \). Let us consider the cut-off function \( \eta_{i2} \) defined in (3.33)-(3.34). Testing (3.1) against \( a(\varepsilon y) \eta_{i2} Z_{ij}, i = 1, \ldots, m \) and \( j = 1, J_i \), we find
\[ \int_{\Omega_p} a(\varepsilon y) \phi \mathcal{L}(\eta_{i2} Z_{ij}) = \int_{\Omega_p} a(\varepsilon y) h \eta_{i2} Z_{ij} + \sum_{k=1}^{J_k} \sum_{t=1}^{J_t} c_{kt} \int_{\Omega_p} \chi_k Z_{kt} \eta_{i2} Z_{ij}. \tag{3.74} \]

Notice that similar to (3.51), by (3.2), (3.14) and (3.57) we can compute that for any \( i = 1, \ldots, m \) and \( j = 1, J_i \),
\[ \Delta_{a(\varepsilon y)} Z_{ij} + \frac{1}{\mu_i^2} \left( 1 + \frac{y - \xi_i^j}{\mu_i} \right)^2 Z_{ij} = O \left( \frac{\varepsilon}{\mu_i^2} \left[ 1 + \frac{|y - \xi_i^j|}{\mu_i} \right]^{-2} \right). \]

Then
\[ \mathcal{L}(\eta_{i2} Z_{ij}) = \eta_{i2} \mathcal{L}(Z_{ij}) - Z_{ij} \Delta_{a(\varepsilon y)} \eta_{i2} = -2 \nabla \eta_{i2} \nabla Z_{ij} \]
\[ = \left[ \frac{1}{\mu_i^2} \left( 1 + \frac{y - \xi_i^j}{\mu_i} \right)^2 - W_{\xi} \right] \eta_{i2} Z_{ij} - \eta_{i2} \left[ \Delta_{a(\varepsilon y)} Z_{ij} + \frac{1}{\mu_i^2} \left( 1 + \frac{y - \xi_i^j}{\mu_i} \right)^2 Z_{ij} \right] + \varepsilon^2 \eta_{i2} Z_{ij} + O \left( \frac{\varepsilon^3}{d^3} \right), \]

\[ \equiv B_{ij} + O \left( \frac{\varepsilon}{\mu_i^2} \left[ 1 + \frac{|y - \xi_i^j|}{\mu_i} \right]^{-2} \right) + O \left( \frac{\varepsilon^2}{\mu_i} \left[ 1 + \frac{|y - \xi_i^j|}{\mu_i} \right]^{-1} \right) + O \left( \frac{\varepsilon^3}{d^3} \right), \]

where
\[ B_{ij} = \left[ \frac{1}{\mu_i^2} \left( 1 + \frac{y - \xi_i^j}{\mu_i} \right)^2 - W_{\xi} \right] \eta_{i2} Z_{ij}. \]

For the estimate of \( B_{ij} \), we split \( \text{supp}(\eta_{i2}) \) into the following pieces:
\[ \widehat{\Omega}_{k1} = \left\{ \frac{z_k}{\mu_k} = \frac{|y - \xi_i^j|}{\mu_k} \leq \frac{1}{3 p^{2k} \varepsilon^{2k}} \right\}, \quad \forall k = 1, \ldots, l, \]
\[ \hat{\Omega}_{k1} = \Omega_p \cap (F^p_k)^{-1} \left( \left\{ \frac{|z_k|}{\mu_k} \leq \frac{1}{3p^{2\kappa} \varepsilon \mu_k} \right\} \cap \mathbb{R}^2_+ \right) \quad \forall \ k = l + 1, \ldots, m, \]

and
\[ \hat{\Omega}_q = (F^p_q)^{-1} \left( \left\{ \frac{|z_q|}{\rho_{0v_0}} \leq \frac{1}{3p^{2\kappa} \varepsilon \rho_{0v_0}} \right\} \cap \mathbb{R}^2_+ \right) \quad \text{and} \quad \hat{\Omega}_2 = \text{supp}(\eta_2) \setminus \bigcup_{k=1}^m \hat{\Omega}_{k1} \cup \hat{\Omega}_q. \]

By (2.3), (3.47) and (3.57) we have that for any \( y \in \hat{\Omega}_q, \)
\[ |y - \xi'_i| \geq |\xi'_i - q'| - |y - q'| \geq |\xi'_i - q'| - \frac{\sqrt{\rho_{0v_0}}}{\varepsilon p^{2\kappa}} > \frac{1}{2\varepsilon p^\kappa}, \quad (3.75) \]
and for any \( y \in \hat{\Omega}_{1k} \) with \( k \neq i, \)
\[ |y - \xi'_i| \geq |\xi'_i - \xi'_k| - |y - \xi'_k| \geq |\xi'_i - \xi'_k| - \frac{\sqrt{\rho_{0v_0}}}{\varepsilon p^{2\kappa}} > \frac{1}{2\varepsilon p^\kappa}. \quad (3.76) \]

In \( \hat{\Omega}_{1i}, \) by using (3.8), (3.14), and the expansion of \( W_{c'} \) in (2.42) we give, for any \( i = 1, \ldots, l \) and \( j = 1, 2, \)
\[ B_{ij} = -\frac{1}{\mu_i^2} \left( \frac{8}{|y - \xi'_i|^2} \right)^2 \mathcal{Z}_j \left( \frac{y - \xi'_i}{\mu_i} \right) \left\{ \frac{1}{p} \left( \tilde{\omega}_1 - V_{1,0} - \frac{1}{2} V_{1,0}^2 \right) \left( \frac{y - \xi'_i}{\mu_i} \right) + O \left( \frac{\log^4 \left( \frac{|y - \xi'_i|}{\mu_i} + 2 \right)}{p^2} \right) \right\}, \]
and for any \( i = l + 1, \ldots, m \) and \( j = 1, \)
\[ B_{ij} = -\frac{1}{\mu_i^2} \left( \frac{8}{|y - \xi'_i|^2} \right)^2 \mathcal{Z}_j \left( \frac{1}{\mu_i} F^p_i (y) \right) \left\{ \frac{1}{p} \left( \tilde{\omega}_1 - V_{1,0} - \frac{1}{2} V_{1,0}^2 \right) \left( \frac{y - \xi'_i}{\mu_i} \right) + O \left( \frac{\log^4 \left( \frac{|y - \xi'_i|}{\mu_i} + 2 \right)}{p^2} \right) \right\}. \]

In \( \hat{\Omega}_q, \) by (3.2), (3.8), (3.14) (3.57), (3.75) and the expansion of \( W_{c'} \) in (2.43),
\[ B_{ij} = \left[ O \left( \frac{\mu_i^2}{|y - \xi'_i|^4} \right) + \left( \frac{\varepsilon}{\rho_{0v_0}} \right)^2 O \left( \frac{8(1 + \alpha)^2 |y - q'|^{2\alpha}}{(1 + |y - q'|^{2(1 + \alpha)^2})^2} \right) \right] O \left( \varepsilon p^\kappa. \right). \]

In \( \hat{\Omega}_{k1} \) with \( k \neq i, \) by (3.2), (3.8), (3.14), (3.57), (3.76) and the expansion of \( W_{c'} \) in (2.42),
\[ B_{ij} = \left[ O \left( \frac{\mu_i^2}{|y - \xi'_i|^4} \right) + O \left( \frac{1}{\mu_k^2} \left( \frac{\rho_{0v_0}}{\mu_k} \right)^{2(1 + \alpha)} \right) \right] O \left( \varepsilon p^\kappa \right). \]

In \( \hat{\Omega}_2, \) by the estimate of \( W_{c'} \) in (2.41),
\[ B_{ij} = \left( \frac{\rho_{0v_0}}{\varepsilon} \right)^{2(1 + \alpha)} O \left( \frac{1}{|y - q'|^{4 + 2\alpha}} \right) + \sum_{k=1}^m O \left( \frac{\mu_k^2}{|y - \xi'_k|^4} \right) \] \[ O \left( \frac{p^{2\kappa} \sqrt{\varepsilon \mu_k}}{\mu_i} \right). \]

We denote that for any \( i = 1, \ldots, l \) and \( j = 1, 2, \)
\[ \hat{\phi}_i (z) = \phi \left( (F^p_i)^{-1} \mu_i z \right), \quad E_j (\hat{\phi}_i) = \int_{B_{\frac{8z_j}{p^{\alpha} \sqrt{\varepsilon \mu_i}}}} \frac{8z_j}{(1 + |z|^2)^2} \hat{\phi}_i \left( \tilde{\omega}_1 - V_{1,0} - \frac{1}{2} V_{1,0}^2 \right) dz, \]
and for any \( i = l + 1, \ldots, m \) and \( j = 1, \)
\[ \hat{\phi}_i (z) = \phi \left( (F^p_i)^{-1} \mu_i z \right), \quad E_j (\hat{\phi}_i) = \int_{B_{\frac{8z_j}{p^{\alpha} \sqrt{\varepsilon \mu_i}}}} \frac{8z_j}{(1 + |z|^2)^3} \hat{\phi}_i \left( \tilde{\omega}_1 - V_{1,0} - \frac{1}{2} V_{1,0}^2 \right) dz. \]

Thus by (3.57),
\[ \int_{\Omega_p} a(\varepsilon y) \phi \mathcal{L}(\eta_2 Z_{ij}) = -\frac{1}{pp_i} a(\xi_i) E_j (\hat{\phi}_i) + O \left( \frac{1}{p^2 \mu_i} |\phi|_{L^\infty (\Omega_p)} \right). \quad (3.77) \]
On the other hand, since \( \| \eta_{i2}Z_{ij} \|_{L^\infty(\Omega_p)} \leq C \mu_i^{-1} \), we obtain
\[
\int_{\Omega_p} a(xy)h \eta_{i2}Z_{ij} = O \left( \frac{1}{\mu_i} \| h \|_* \right). 
\]
(3.78)
Moreover, by (3.2), (3.8), (3.14), (3.57) and (3.75) we conclude that if \( 1 \leq k = i \leq l \),
\[
\int_{\Omega_p} \chi_kZ_{kt} \eta_{i2}Z_{ij} = \int_{\mathbb{R}^2} \chi \sum_{j=1}^l Z_j dZ = D_i \delta_{ij}, 
\]
(3.79)
and if \( l + 1 \leq k = i \leq m \),
\[
\int_{\Omega_p} \chi_kZ_{kt} \eta_{i2}Z_{ik} = \int_{\mathbb{R}^2} \chi \sum_{j=1}^l \mu_i |Z_j| dZ = \frac{1}{2}D_1 \left( 1 + O(\varepsilon \mu_i) \right), 
\]
(3.80)
while if \( k \neq i \),
\[
\int_{\Omega_p} \chi_kZ_{kt} \eta_{i2}Z_{ij} = O(\varepsilon \mu_k p^\ast). 
\]
(3.81)
As a consequence, substituting estimates (3.77)-(3.81) into (3.74), we have that for any \( i = 1, \ldots, m \) and \( j = 1, J_i \),
\[
D_j c_{ij} + O \left( \sum_{k=1}^{J_i} \sum_{t=1}^{J_k} \varepsilon \mu_k p^\ast |c_{kt}| \right) = O \left( \frac{1}{\mu_i} \| h \|_* + \frac{1}{p \mu_i} \| \phi \|_{L^\infty(\Omega_p)} \right), 
\]
and then, by (2.28),
\[
\sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| = O \left( \| h \|_* + \frac{1}{p} \| \phi \|_{L^\infty(\Omega_p)} \right). 
\]
(3.82)
Since \( \sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| = o(1) \), as in contradiction arguments of Step 2, we deduce that for any \( i = 1, \ldots, l \),
\[
\tilde{\phi}_i \to C_i \left( \frac{|z|^2 - 1}{|z|^2 + 1} \right) \text{ uniformly in } C^0_{\text{loc}}(\mathbb{R}^2), 
\]
but for any \( i = l + 1, \ldots, m \),
\[
\tilde{\phi}_i \to C_i \left( \frac{|z|^2 - 1}{|z|^2 + 1} \right) \text{ uniformly in } C^0_{\text{loc}}(\mathbb{R}^2_+), 
\]
with some constant \( C_i \). Hence we have a more delicate estimate in (3.77), because by Lebesgue’s theorem we find that for any \( i = 1, \ldots, l \) and \( j = 1, 2 \),
\[
E_j(\tilde{\phi}_i) \to C_i \int_{\mathbb{R}^2} \frac{8z_j}{(|z|^2 + 1)^3} \frac{|z|^2 - 1}{|z|^2 + 1} \left( \tilde{\phi}_1 - V_{1,0} - \frac{1}{2}V_{1,0}^2 \right) (|z|) dz = 0, 
\]
and for any \( i = l + 1, \ldots, m \) and \( j = 1 \),
\[
E_j(\tilde{\phi}_i) \to C_i \int_{\mathbb{R}^2} \frac{8z_j}{(|z|^2 + 1)^3} \frac{|z|^2 - 1}{|z|^2 + 1} \left( \tilde{\phi}_1 - V_{1,0} - \frac{1}{2}V_{1,0}^2 \right) (|z|) dz = 0. 
\]
Therefore,
\[
\sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i |c_{ij}| = o \left( \frac{1}{p} \right) + O \left( \| h \|_* \right), 
\]
which contradicts (3.73). So estimate (3.72) is established and then by (3.82), we obtain
\[
|c_{ij}| \leq C \frac{1}{\mu_i} \| h \|_*. 
\]
Now consider the Hilbert space
\[ H_\xi = \left\{ \phi \in H^1(\Omega_p) \mid \int_{\Omega_p} \chi_i Z_{ij} \phi = 0 \quad \text{for any } i = 1, \ldots, m, \ j = 1, J_i; \ \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial \Omega_p \right\} \]
with the norm \( \| \phi \|^2_{H^1(\Omega_p)} = \int_{\Omega_p} a(\varepsilon y)(|\nabla \phi|^2 + \varepsilon^2 \phi^2). \) Equation (3.1) is equivalent to find \( \phi \in H_\xi \) such that
\[ \int_{\Omega_p} a(\varepsilon y)(|\nabla \phi|^2 + \varepsilon^2 \phi^2) - \int_{\Omega_p} a(\varepsilon y)W(\phi) = \int_{\Omega_p} a(\varepsilon y)h \psi \quad \forall \psi \in H_\xi. \]
By Fredholm’s alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by estimate (3.72). Finally, for \( p \geq p_m \) fixed, by density of \( C_0^\infty(\overline{\Omega}_p) \) in \( (C(\overline{\Omega}_p), \| \cdot \|_{L^\infty(\Omega_p)}) \), we can approximate \( h \in C(\overline{\Omega}_p) \) by smooth functions and, by (3.72) and elliptic regularity theory, we find that for any \( h \in C(\overline{\Omega}_p) \), problem (3.1) admits a unique solution which belongs to \( H^2(\Omega_p) \) and satisfies the a priori estimate (3.15). \( \square \)

**Remark 3.5.** Given \( h \in C(\overline{\Omega}_p) \) with \( \| h \|_\ast < \infty \), let \( \phi \) be the solution to equation (3.1) given by Proposition 3.1. Testing (3.1) against \( a(\varepsilon y)\phi \), we obtain
\[ \| \phi \|^2_{H^1(\Omega_p)} = \int_{\Omega_p} a(\varepsilon y)W(\phi)^2 + \int_{\Omega_p} a(\varepsilon y)h \phi, \]
and then, by (2.41),
\[ \| \phi \|_{H^1(\Omega_p)} \leq C(\| h \|_\ast + \| \phi \|_{L^\infty(\Omega_p)}). \]

Now let us consider the nonlinear problem: for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \), we find a function \( \phi \) and scalars \( c_{ij}, i = 1, \ldots, m, j = 1, J_i \) such that
\[ \begin{align*}
\mathcal{L}(\phi) &= -[R(\phi) + N(\phi)] + \frac{1}{a(\varepsilon y)} \sum_{i=1}^m \sum_{j=1}^{J_i} c_{ij} \chi_i Z_{ij} \quad \text{in } \Omega_p, \\
\frac{\partial \phi}{\partial \nu} &= 0 \quad \text{on } \partial \Omega_p, \\
\int_{\Omega_p} \chi_i Z_{ij} \phi &= 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, J_i. 
\end{align*} \tag{3.83} \]

**Proposition 3.6.** Let \( q \in \mathcal{T} \) and \( m \) be a non-negative integer. Then there exist constants \( C > 0 \) and \( p_m > 1 \) such that for any \( p \geq p_m \) and any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \), problem (3.83) admits a unique solution \( \phi_\xi \), for some coefficients \( c_{ij}(\xi^i), i = 1, \ldots, m, j = 1, J_i \), such that
\[ \| \phi_\xi \|_{L^\infty(\Omega_p)} \leq \frac{C}{p^\frac{1}{2}}, \quad \sum_{i=1}^m \sum_{j=1}^{J_i} \mu_i|c_{ij}(\xi^i)| \leq \frac{C}{p^\frac{1}{2}} \quad \text{and} \quad \| \phi_\xi \|_{H^1(\Omega_p)} \leq \frac{C}{p^3}. \tag{3.84} \]
Furthermore, the map \( \xi^i \mapsto \phi_\xi \) is a \( C^1 \)-function in \( C(\overline{\Omega}_p) \) and \( H^1(\Omega_p) \).

**Proof.** Proposition 3.1, Remarks 2.5 and 3.5 allow us to apply the contraction mapping theorem and the implicit function theorem to find a solution for problem (3.83) satisfying (3.84) and the corresponding regularity of the map \( \xi^i \mapsto \phi_\xi \). Since it is a standard procedure, we omit the detailed proof here. \( \square \)

**Remark 3.7.** The function \( V(\xi) + \phi_\xi \), where \( \phi_\xi \) is the unique solution of problem (3.83) given by Proposition 3.6, is positive in \( \overline{\Omega}_p \). In fact, we notice that \( p^2 \phi_\xi \to 0 \) uniformly over \( \overline{\Omega}_p \). Furthermore, in the region \( |y - q^i| \geq 1/(\varepsilon p^{2\kappa}) \) and \( |y - \xi^i| \geq 1/(\varepsilon p^{2\kappa}) \) for each \( i = 1, \ldots, m \), by (2.5), (2.23) and the definition of \( V(\xi) \) in (2.34) we can derive that \( V(\xi) + \phi_\xi \) is positive. Outside this region, we may conclude the same result from Remark 2.3.
4. Variational reduction

Since problem (3.83) has been solved, we just find a solution of problem (2.38) with $m \geq 1$ and hence to the original problem (1.1) if we find $\xi'$ such that the coefficient $c_{ij}(\xi')$ in (3.83) satisfies

$$c_{ij}(\xi') = 0 \quad \text{for all } i = 1, \ldots, m, \ j = 1, J.$$  \hspace{1cm} (4.1)

Let us consider the energy function $J_p$ associated to problem (1.1), namely

$$J_p(u) = \frac{1}{2} \int_\Omega a(x)(|\nabla u|^2 + u^2)dx - \frac{1}{p+1} \int_\Omega a(x)|x - q|^{2a}u^{p+1}dx, \quad u \in H^1(\Omega).$$  \hspace{1cm} (4.2)

For any integer $m \geq 1$, we can introduce the reduced energy

$$F_p(\xi) = J_p(U_\xi + \tilde{\phi}_\xi), \quad \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q),$$  \hspace{1cm} (4.3)

where $U_\xi$ is our approximate solution defined in (2.18) and

$$\tilde{\phi}_\xi(x) = \varepsilon^{-2/(p-1)}\phi_\xi(e^{-1}x), \quad x \in \Omega,$$  \hspace{1cm} (4.4)

with $\phi_\xi$ the unique solution to problem (3.83) given by Proposition 3.6. Then we obtain that critical points of $F_p$ correspond to solutions of (4.1) for large $p$. That is:

**Proposition 4.1.** For any integer $m \geq 1$, the function $F_p : \mathcal{O}_p(q) \mapsto \mathbb{R}$ is of class $C^1$. Moreover, for all $p$ sufficiently large, if $D_\xi F_p(\xi) = 0$, then $\xi' = \xi/\varepsilon$ satisfies (4.1).

**Proof.** From the result obtained in Proposition 3.6 and the definition of function $U_\xi$ we have clearly that for any integer $m \geq 1$, the function $F_p : \mathcal{O}_p(q) \mapsto \mathbb{R}$ is of class $C^1$ since the map $\xi \mapsto \tilde{\phi}_\xi$ is a $C^1$-map into $H^1(\Omega)$.

Recalling the definition of $I_p$ in (2.35) and making a change of variable, we give

$$F_p(\xi) = J_p(U_\xi + \tilde{\phi}_\xi) = \varepsilon^{-4/(p-1)}I_p(V_{\xi'} + \phi_\xi).$$  \hspace{1cm} (4.5)

Assume that $\phi_\xi$ solves problem (3.83) and $D_\xi F_p(\xi) = 0$. Then we have that for any $k = 1, \ldots, m$ and $t = 1, J_k$,

$$0 = I'_p(V_{\xi'} + \phi_\xi) \partial_{(\xi')_k}(V_{\xi'} + \phi_\xi) = \sum_{i=1}^m \sum_{j=1}^{J_i} c_{ij}(\xi') \int_{\Omega_p} \chi_i Z_{ij} \partial_{(\xi')_k} V_{\xi'} - \sum_{i=1}^m \sum_{j=1}^{J_i} c_{ij}(\xi') \int_{\Omega_p} \phi_{\xi_k} \partial_{(\xi')_k}(\chi_i Z_{ij}).$$  \hspace{1cm} (4.6)

Note that $V_{\xi'}(y) = \varepsilon^{2/(p-1)}U_\xi(\varepsilon y)$. From (2.23), (2.24), (2.26) and the definition of $U_\xi$ in (2.18), we obtain

$$\partial_{(\xi')_k} V_{\xi'}(y) = \sum_{i=1}^m \frac{\varepsilon^{2/(p-1)}|\xi_i - q|^{2a/(p-1)}}{\gamma \mu_i^{2/(p-1)}} \left\{ \partial_{(\xi')_k} \left[ V_{\xi_i, \xi}(\varepsilon y) + \frac{1}{p} \omega_1 \left( \frac{\varepsilon y - \xi_i}{p\mu_i} \right) + \frac{1}{p^2} \omega_2 \left( \frac{\varepsilon y - \xi_i}{p\mu_i} \right) \right] \ight. \left. + \gamma \mu_i^{2/(p-1)}|\xi_i - q|^{2a/(p-1)} H_0(\varepsilon y) \right\} - \frac{2 \partial_{(\xi')_k} \log(\mu_i)}{p-1} \left( \frac{\varepsilon y - \xi_i}{p\mu_i} \right) - \frac{2 \partial_{(\xi')_k} \log(\mu_i)}{p-1} \left( \frac{\varepsilon y - \xi_i}{p\mu_i} \right) - \frac{2 \partial_{(\xi')_k} \log(\mu_i)}{p-1} \left( \frac{\varepsilon y - \xi_i}{p\mu_i} \right).$$

Using the fact that $|\partial_{(\xi')_k} \log(\mu_i)| = O(\varepsilon p^k)$ for any $i = 0, 1, \ldots, m$, we have that by (2.1), (2.5) and (3.2),

$$\partial_{(\xi')_k} U_{\delta_0}(\varepsilon y - q) = O(\varepsilon p^k), \quad \partial_{(\xi')_k} V_{\delta_0, \xi}(\varepsilon y) = \frac{4 \delta_{ki} \chi_i}{\mu_i} \left( \frac{y - \xi_i'}{\mu_i} \right) + O(\varepsilon p^k).$$
and for each \( j = 1, 2 \), by (2.12)-(2.13),

\[
\partial_{(\xi_j^i)} \omega_j \left( \frac{ey - q}{\rho_0 v_0} \right) = O(\varepsilon p^\kappa), \quad \partial_{(\xi_j^i)} \tilde{\omega}_j \left( \frac{y - \xi_j^i}{\mu_i} \right) = \frac{\delta_{ki}}{\mu_i} \left[ \tilde{G}_j Z_i \left( \frac{y - \xi_j^i}{\mu_i} \right) + O \left( \frac{\mu_i^2}{y - \xi_j^i} \right) \right] + O(\varepsilon p^\kappa),
\]

where \( \delta_{ki} \) denotes the Kronecker’s symbol. Moreover, similar to the proof in Lemma 2.1, we can prove that

\[
\frac{\delta_{ki}}{\mu_i} \left[ \tilde{G}_j Z_i \left( \frac{y - \xi_j^i}{\mu_i} \right) + O \left( \frac{\mu_i^2}{y - \xi_j^i} \right) \right] + O(\varepsilon p^\kappa),
\]

Then

\[
\partial_{(\xi_j^i)} V_{y^i}(y) = \frac{\varepsilon^{2/2(p-1)}(\xi_j^i)}{\gamma \mu_k^{2/2(p-1)}|\xi_k - q|^{4\alpha/(p-1)}} \left\{ \frac{4}{\mu_k} Z_i \left( \frac{y - \xi_j^i}{\mu_k} \right) + O \left( \frac{1}{p \mu_k} \right) \right\} + O(\varepsilon p^{\kappa-1}). \tag{4.7}
\]

On the other hand, by (3.2), (3.8), (3.14) and (3.57) we can compute

\[
|\partial_{(\xi_j^i)} (\chi_i Z_{ij})| = O \left( \frac{1}{\mu_i} \varepsilon p^\kappa + \frac{1}{\mu_i} \delta_{ki} \right). \tag{4.8}
\]

Consequently, (4.6) can be written as, for each \( k = 1, \ldots, m \) and \( t = 1, J_k \),

\[
\sum_{i,j} \frac{\varepsilon^{2/2(p-1)} c_{ij}(\xi_j^i)}{\gamma \mu_k^{2/2(p-1)}|\xi_k - q|^{4\alpha/(p-1)}} \left\{ \delta_{ki} \left[ \frac{c_i}{2 \pi} \int_{\mathbb{R}^2} \chi Z_i Z_t + O \left( \frac{1}{p} \right) \right] + (1 - \delta_{ki})O \left( \frac{1}{\mu_i} \right) \right\}
\]

\[
+ \sum_{ij} |c_{ij}(\xi_j^i)| \{ \left( O(\mu_i \varepsilon p^{\kappa-1}) + \| \phi_{y^i} \|_{L^\infty(\Omega)} + O(\mu_i \varepsilon p^\kappa + \delta_{ki}) \right) \} = 0,
\]

and then, by (2.3), (2.5), (2.28) and (3.84),

\[
\frac{c_k c_k(\xi_j^i)}{p^{2/2(p-1)} \mu_k^{2/2(p-1)}|\xi_k - q|^{4\alpha/(p-1)}} \int_0^{R_0+1} \frac{\chi(r) r^3}{(1 + r^2)^2} dr + \sum_{i=1}^m \sum_{j=1}^J |c_{ij}(\xi_j^i)| \{ O \left( \frac{\delta_{ki} + 2}{p^2} + \varepsilon \mu_i p^{\kappa-1} + \varepsilon p^\kappa \right) \} = 0,
\]

which implies \( c_k(\xi_j^i) = 0 \) for each \( k = 1, \ldots, m \) and \( t = 1, J_k \). \qed

Moreover, in order to solve for critical points of \( F_p \), we need to give the following reduced energy expansion.

**Proposition 4.2.** Let \( q \in \Omega \) and \( m \) be a positive integer. With the choices for the parameters \( \mu_0 \) and \( \mu_i \), \( i = 1, \ldots, m \), respectively given by (2.25) and (2.27), there exists \( p_m > 1 \) such that for any \( p > p_m \) and any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \), the following expansion uniformly holds

\[
F_p(\xi) = \frac{e}{2 p} \sum_{i=1}^m c_i a(\xi_i) \left\{ 1 + \frac{\bar{K}}{p} + \frac{2 \log p}{p} - \frac{4 \alpha \log |\xi_i - q|}{p} - \frac{1}{p} \left[ c_i H(\xi_i, \xi_i) + c_0 G(\xi_i, q) + \sum_{k \neq i} c_k G(\xi_i, \xi_k) \right] \right\}
\]

\[
+ \frac{e}{2 p} c_0 a(q) \left\{ 1 + \frac{\bar{K}}{p} + \frac{2 \log p}{p} - \frac{1}{p} \left[ c_0 H(q, q) + \sum_{k=1}^{m} c_k G(q, \xi_k) \right] \right\} + O \left( \frac{\log^2 p}{p^3} \right), \tag{4.9}
\]

where the coefficients \( c_0 \) and \( c_k \), \( k = 1, \ldots, m \), are defined in (2.20) and

\[
\bar{K} = \frac{1}{8 \pi (1 + \alpha)} \int_{\mathbb{R}^2} \frac{8(1 + \alpha)^2 |z|^{2 \alpha}}{u_0(z) - \Delta \omega_1(z)} \, dz, \quad \bar{K} = \frac{1}{8 \pi} \int_{\mathbb{R}^2} \frac{8}{(1 + |z|^2)^2} V_{1,0}(\hat{z}) - \Delta \tilde{\omega}_1(\hat{z}) \, d\hat{z}.
\]

**Proof.** First of all, multiply the first equation in (3.83) by \( a(\xi y) (V_{y^i} + \phi_{y^i}) \) and integrate by parts to give

\[
\int_{\Omega_p} a(\xi y) |\nabla (V_{y^i} + \phi_{y^i})|^2 + \varepsilon^2 (V_{y^i} + \phi_{y^i})^2 \right\}
\]

\[
= \int_{\Omega_p} a(\xi y) |\xi y - q|^{2\alpha} (V_{y^i} + \phi_{y^i})^{p+1} + \sum_{i=1}^m \sum_{j=1}^J c_{ij}(\xi_j^i) \int_{\Omega_p} \chi_i Z_{ij} V_{y^i}.
\]
Since \( V_\epsilon \) is a uniformly bounded function, by (3.84) we get
\[
\int_{\Omega_p} a(\epsilon y) \left[ |\nabla (V_\epsilon + \phi_\epsilon)|^2 + \epsilon^2 (V_\epsilon + \phi_\epsilon)^2 \right] = \int_{\Omega_p} a(\epsilon y) |\epsilon y - q|^{2\alpha} (V_\epsilon + \phi_\epsilon)^{p+1} + O \left( \frac{1}{p^4} \right)
\]
uniformly for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(q) \). Then by (2.35) and (4.5) we have
\[
F_p(\xi) = \left( \frac{2}{2 - \frac{1}{p + 1}} \right) \epsilon^{-4/(p-1)} \int_{\Omega_p} a(\epsilon y) \left[ |\nabla (V_\epsilon + \phi_\epsilon)|^2 + \epsilon^2 (V_\epsilon + \phi_\epsilon)^2 \right] dy + O \left( \frac{1}{p^4} \right)
\]
\[
= \left( \frac{2}{2 - \frac{1}{p + 1}} \right) \epsilon^{-4/(p-1)} \left\{ \int_{\Omega_p} a(\epsilon y) \left( |\nabla V_\epsilon|^2 + \epsilon^2 V_\epsilon^2 \right) dy + 2 \int_{\Omega_p} a(\epsilon y) \left( \nabla V_\epsilon \cdot \nabla \phi_\epsilon + \epsilon^2 V_\epsilon \phi_\epsilon \right) dy + \int_{\Omega_p} a(\epsilon y) \left( |\nabla \phi_\epsilon|^2 + \epsilon^2 \phi_\epsilon^2 \right) dy \right\} + O \left( \frac{1}{p^4} \right)
\]
\[
= \left( \frac{2}{2 - \frac{1}{p + 1}} \right) \int_{\Omega_p} a(x) \left( |\nabla U_\epsilon|^2 + U_\epsilon^2 \right) dx + O \left( \frac{1}{p^4} \int_{\Omega} a(x) \left( |\nabla U_\epsilon|^2 + U_\epsilon^2 \right) dx \right)^{1/2} + O \left( \frac{1}{p^5} \right).
\]
Let us expand the leading term \( \int_{\Omega} a(x) \left( |\nabla U_\epsilon|^2 + U_\epsilon^2 \right) dx \): in view of (2.5)-(2.7), (2.18)-(2.20), (2.23)-(2.24) and (2.26) we obtain
\[
\int_{\Omega} a(x) \left( |\nabla U_\epsilon|^2 + U_\epsilon^2 \right) dx = \int_{\Omega} a(x) (-\Delta u U_\epsilon + U_\epsilon) U_\epsilon dx
\]
\[
= \sum_{i=1}^{m} \frac{1}{\gamma_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \int_{\Omega \cap B \frac{1}{p^{\alpha}} |\xi_i - q|^{1/(p-1)}} a(x) \left[ \delta_i^2 \left( \frac{V_{1,0}(z)}{\delta_i} \right) - \frac{1}{p} \Delta \delta_i \left( \frac{x - \xi_i}{\delta_i} \right) - \frac{1}{p^2} \Delta \delta_i \left( \frac{x - \xi_i}{\delta_i} \right) \right] U_\epsilon(x) dx
\]
\[
+ \frac{1}{\gamma_0^{2/(p-1)}} \int_{\Omega \cap B \frac{1}{p^{\alpha}} |\xi_0|} a(x) \left[ \frac{x - q}{\delta_0} \right] \epsilon U_1(x) \left( \frac{x - q}{\delta_0} \right) - \frac{1}{p} \Delta \omega_1 \left( \frac{x - q}{\delta_0} \right) - \frac{1}{p^2} \Delta \omega_1 \left( \frac{x - q}{\delta_0} \right) \right] U_\epsilon(x) dx + O \left( \frac{1}{p^5} \right)
\]
\[
= \sum_{i=1}^{m} \frac{1}{\gamma_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \left[ p + \frac{V_{1,0}(z)}{\delta_i} + \frac{1}{p^2} \omega_2(z) + O \left( \frac{\delta_i^2 |z|^\beta + p^{2\alpha+1}\delta_i^{1+\alpha}}{1 + |z|^{2(1+\alpha)}} + \sum_{k=0}^{\infty} \delta_k^{\beta/2} \right) \right] dz + O \left( \frac{1}{p^5} \right)
\]
\[
= \sum_{i=1}^{m} \frac{c_i a(\xi_i)}{\gamma_i^{2/(p-1)} |\xi_i - q|^{2\alpha/(p-1)}} \left[ p + \frac{\bar{K}}{p} + O \left( \frac{1}{p} \right) \right] + \frac{c_0 a(q)}{\gamma_0^{2/(p-1)}} \left[ p + \frac{K}{p} + O \left( \frac{1}{p} \right) \right] + O \left( \frac{1}{p^5} \right)
\]
Recalling that \( \gamma = p^{p/(p-1)} e^{-p/(2p-2)} \), we find
\[
\frac{1}{\gamma_i^2} = \frac{e}{p^2} \left[ 1 - \frac{2 \log p}{p} + O \left( \frac{\log^2 p}{p^2} \right) \right], \quad \frac{1}{\mu_i^{4/(p-1)}} = 1 - \frac{4 \log \mu_0}{p} + O \left( \frac{\log^2 \mu_0}{p^2} \right),
\]
and for each \( i = 1, \ldots, m \),
\[
\frac{1}{\mu_i^{4/(p-1)} |\xi_i - q|^{4\alpha/(p-1)}} = 1 - \frac{4 \log \left( \mu_i |\xi_i - q|^{\alpha} \right)}{p} + O \left( \frac{\log^2 \mu_i + \log^2 |\xi_i - q|}{p^2} \right).
\]
Thus by (2.3) and (2.28),
\[
\frac{1}{\gamma^2 \mu_0^{4/(p-1)}} = \frac{1}{p^2} \left[ 1 - \frac{2 \log \gamma}{p} \right],
\]
\[
\frac{1}{\gamma^2 \mu_i^{4/(p-1)}} = \frac{1}{p^2} \left[ 1 - \frac{2 \log \gamma}{p} \right],
\]
Hence
\[
F_p(\xi) = \frac{e}{2p} \sum_{i=1}^{m} c_i a(\xi_i) \left( 1 - \frac{2 \log \gamma}{p} \right) + \frac{e}{2p} c_0 a(\xi) \left( 1 - \frac{2 \log \gamma}{p} \right),
\]
which, together with the expansions of $\mu_0, \mu_i$ in (2.29)-(2.30), implies that expansion (4.9) holds. \qed

5. Proofs of Theorems

Proof of Theorem 1.1. According to Proposition 4.1, the function $u_p = U_\xi + \tilde{\phi}_\xi$ is a solution to problem (1.1) if we adjust $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(\xi)$ with $q \in \Omega$ so that it is a critical point of $F_p$ defined in (4.3). For this aim, let us claim that for any integer $m \geq 1$ and any $p > 1$ large enough, the maximization problem
\[
\max_{(\xi_1, \ldots, \xi_m) \in \mathcal{O}_p(\xi)} F_p(\xi_1, \ldots, \xi_m)
\]
has a solution $\xi^p = (\xi_1^p, \ldots, \xi_m^p) \in \mathcal{O}_p(\xi)$, i.e., the interior of $\mathcal{O}_p(\xi)$. Once this claim is proven, we can easily get the qualitative properties of solutions of problem (1.1) as predicted in Theorem 1.1.

Let $\xi^p = (\xi_1^p, \ldots, \xi_m^p)$ be the maximizer of $F_p$ over $\mathcal{O}_p(\xi)$. We need to prove that $\xi^p$ belongs to the interior of $\mathcal{O}_p(\xi)$. First, we obtain a lower bound for $F_p$ over $\mathcal{O}_p(\xi)$. Let
\[
\hat{\xi}_i = q + \frac{1}{\sqrt{p}}, \quad i = 1, \ldots, m,
\]
where $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_m)$ forms a $m$-regular polygon in $\mathbb{R}^2$. Obviously, $\xi^0 = (\xi_1^0, \ldots, \xi_m^0) \in \mathcal{O}_p(\xi)$ since $q \in \Omega$ and $\kappa > 1$. Using (4.9) and the fact that $q \in \Omega$ is a strict local maximum point of $a(x)$, we obtain
\[
\max_{\hat{\xi} \in \mathcal{O}_p(\xi)} F_p(\xi) \geq F_p(\hat{\xi}) \geq \frac{e}{2p^2} \left[ 8 \pi (m + 1 + \alpha) a(q) - 16 \pi (m + 1) (m + 1 + \alpha) a(q) \log p + O(1) \right]. \tag{5.1}
\]
Next, we suppose $\xi^p = (\xi_1^p, \ldots, \xi_m^p) \in \partial \mathcal{O}_p(\xi)$. Then there exist three possibilities:

C1. There exists an $i_0$ such that $\xi_{i_0}^p \in \partial B_d(q)$, in which case, $a(\xi_{i_0}^p) < a(q) - d_0$ for some $d_0 > 0$ independent of $p$;

C2. There exist indices $i_0, j_0, i_0 \neq j_0$ such that $|\xi_{i_0}^p - \xi_{j_0}^p| = p^{-\kappa}$;

C3. There exists an $k_0$ such that $|\xi_{k_0}^p - q| = p^{-\kappa}$.

For the first case, we have
\[
\max_{\hat{\xi} \in \mathcal{O}_p(\xi)} F_p(\xi) \leq \frac{e}{2p^2} \left[ 8 \pi p ((m + 1 + \alpha) a(q) - d_0) + O(\log p) \right], \tag{5.2}
\]
which contradicts to (5.1). This shows that $a(\xi_{i_0}^p) \to a(q)$. By the condition over $a$, we get $\xi_{i_0}^p \to q$ for all $i = 1, \ldots, m$.

For the second case, we have
\[
\max_{\hat{\xi} \in \mathcal{O}_p(\xi)} F_p(\xi) \leq \frac{e}{2p^2} \left[ 8 \pi (m + 1 + \alpha) (p - 2 \log p) a(q) + 32 \pi (a(\xi_{i_0}^p) + a(\xi_{j_0}^p)) \log |\xi_{i_0}^p - \xi_{j_0}^p| + O(1) \right]
\]
\[
\leq \frac{e}{2p^2} \left[ 8 \pi (m + 1 + \alpha) (p - 2 \log p) a(q) - 32 \pi \kappa (a(\xi_{i_0}^p) + a(\xi_{j_0}^p)) \log p + O(1) \right]. \tag{5.3}
\]
For the last case, we have
\[
\max_{\xi \in \mathcal{B}(q)} F_p(\xi) \leq \frac{e}{2p^2} \left\{ 8\pi (m+1+\alpha)(p-2\log p) a(q) + 32\pi \left[ a(\xi_{k_0}^p) + (1+\alpha)a(q) \right] \log |\xi_{k_0}^p - q| + O(1) \right\}
\]
\[
\leq \frac{e}{2p^2} \left\{ 8\pi (m+1+\alpha)(p-2\log p) a(q) - 32\pi \left[ a(\xi_{k_0}^p) + (1+\alpha)a(q) \right] \log p + O(1) \right\}. \tag{5.4}
\]
Combining (5.3)-(5.4) with (5.1), we find
\[
32\pi \kappa \max \left\{ a(\xi_{k_0}^p), a(\xi_{j_0}^p) \right\} \log p \leq 16\pi (m+1+\alpha)a(q) \log p + O(1), \tag{5.5}
\]
which is impossible by the choice of \( \kappa \) in (2.4).
\[
\square
\]
Proof of Theorem 1.2. By Proposition 4.1 we have to find a critical point \( \xi^p = (\xi_1^p, \ldots, \xi_m^p) \in (B_d(q) \cap \Omega)^{m-1} \) of \( F_p \) such that points \( \xi_1^p, \ldots, \xi_m^p \) accumulate to the boundary point \( q \). From (1.9), (2.20), (4.9), Lemma 1.1 and the fact that \( a(q)G(q, \xi_i) = a(\xi_i)G(\xi_i, q) \) and \( a(\xi_i)G(\xi_i, \xi_k) = a(\xi_k)G(\xi_k, \xi_i) \) for all \( i, k = 1, \ldots, m \) with \( i \neq k \), we have that \( F_p \) reduces to
\[
F_p(\xi) = \frac{e}{2p^2} \left\{ 8\pi \sum_{i=1}^l a(\xi_i) \left[ p - 2\log p - 4\alpha \log |\xi_i - q| - 8\pi H(\xi_i, \xi_i) - 8\pi \sum_{k=1, k \neq i} G(\xi_i, \xi_k) \right] \right. \\
-64\pi^2 \sum_{i=1, k=l+1}^m a(\xi_k)G(\xi_k, \xi_i) + 4\pi \sum_{i=l+1}^m a(\xi_i) \left[ p - 2\log p - 4\alpha \log |\xi_i - q| + 4 \sum_{k=l+1, k \neq i} \log |\xi_i - \xi_k| \right] \\
+8\pi (1+\alpha)a(q) \left[ p - 2\log p - 16\pi \sum_{i=1}^l G(q, \xi_i) + 8 \sum_{i=l+1}^m \log |\xi_i - q| \right] + O(1) \tag{5.6}
\]
\( C^0 \)-uniformly in \( \mathcal{O}(q) \). Here we claim that for any integers \( m \geq 1, 0 \leq l \leq m \) and any \( p > 1 \) large enough, the maximization problem
\[
\max_{(\xi_1, \ldots, \xi_m) \in \mathcal{O}(q)} F_p(\xi_1, \ldots, \xi_m)
\]
has a solution \( \xi^p = (\xi_1^p, \ldots, \xi_m^p) \in \mathcal{O}(q) \), i.e., the interior of \( \mathcal{O}(q) \). Once this claim is proven, we can easily get the qualitative properties of solutions of problem (1.1) as predicted in Theorem 1.2.

Let \( \xi^p = (\xi_1^p, \ldots, \xi_m^p) \) be the maximizer of \( F_p \) over \( \mathcal{O}(q) \). We need to prove that \( \xi^p \) lies in the interior of \( \mathcal{O}(q) \). First, we obtain a lower bound for \( F_p \) over \( \mathcal{O}(q) \). Let us consider a smooth change of variables
\[
H_q^p(y) = e^{p/2} H_q(e^{-p/2} y),
\]
where \( H_q : B_d(q) \to \mathcal{M} \) is a diffeomorphism and \( \mathcal{M} \) is an open neighborhood of the origin such that \( H_q(B_d(q) \cap \Omega) = \mathcal{M} \cap \mathbb{R}^2_+ \) and \( H_q(B_d(q) \cap \partial \Omega) = \mathcal{M} \cap \partial \mathbb{R}^2_+ \). Let
\[
\xi_0^0 = q - \frac{t_i}{\sqrt{p}} \nu(q), \quad i = 1, \ldots, l, \quad \text{but} \quad \xi_i^0 = e^{-p/2} (H_q^p)^{-1} \left( \frac{e^{p/2}}{\sqrt{p}} \xi_i^0 \right), \quad i = l + 1, \ldots, m,
\]
where \( t_i > 0 \) and \( \xi_0^0 \in \mathcal{M} \cap \partial \mathbb{R}^2_+ \) satisfy \( t_{i+1} - t_i = \sigma \), \( \xi_0^0 - \xi_{i+1}^0 = \sigma \) for all \( \sigma > 0 \) sufficiently small, fixed and independent of \( p \). Using the expansion \( (H_q^p)^{-1}(z) = e^{p/2} z + O(e^{-p/2} |z|^2) \), we find
\[
\xi_i^0 = q + \frac{1}{\sqrt{p}} \xi_i^0 + O \left( \frac{e^{-p/2}}{\sqrt{p}} |\xi_i^0| \right), \quad i = l + 1, \ldots, m.
\]
Clearly, \( \xi^0 = (\xi_1^0, \ldots, \xi_m^0) \in \mathcal{O}(q) \) because of \( q \in \partial \Omega \) and \( \kappa > 1 \). Notice that \( q \in \partial \Omega \) is a strict local maximum point of \( a(\xi) \) over \( \mathcal{M} \) and satisfies \( \partial_p a(q) = (\nabla a(q), \nu(q)) = 0 \). Then we can derive that there is a constant \( C > 0 \) independent of \( p \) such that
\[
a(q) - \frac{C}{p} \leq a(\xi_i^0) < a(q), \quad i = 1, \ldots, m.
\]
From definition (1.9), Lemmas A.2 and A.3 we conclude that for any $i = 1, \ldots, l$ and $k = 1, \ldots, m$ with $i \neq k$,

$$H(\xi_i^0, \xi_i^0) = \frac{1}{4\pi} \log p + O(1), \quad G(q, \xi_i^0) = \frac{1}{2\pi} \log p + O(1), \quad G(\xi_k^0, \xi_i^0) = \frac{1}{2\pi} \log p + O(1).$$

By (5.6) we find

$$\max_{\xi \in \partial O_p(q)} F_p(\xi) \geq F_p(\xi^0) \geq \frac{2\pi\xi a(q)}{p^2} \left\{ (m + l + 2 + 2\alpha)p - [2(m + l)^2 + (8 + 6\alpha)(m + l) + 4 + 4\alpha] \log p + O(1) \right\}. \quad (5.7)$$

Next, we suppose $\xi^p = (\xi^p_1, \ldots, \xi^p_m) \in \partial O_p(q)$. Then there exist five cases:

C1. There exists an $i_0 \in \{1, \ldots, l\}$ such that $\xi^p_{i_0} \in \partial B_d(q) \cap \Omega$, in which case, $a(\xi^p_{i_0}) < a(q) - d_0$ for some $d_0 > 0$ independent of $p$;

C2. There exists an $i_0 \in \{l + 1, \ldots, m\}$ such that $\xi^p_{i_0} \in \partial B_d(q) \cap \partial \Omega$, in which case, $a(\xi^p_{i_0}) < a(q) - d_0$ for some $d_0 > 0$ independent of $p$;

C3. There exists an $i_0 \in \{1, \ldots, l\}$ such that $\text{dist}(\xi^p_{i_0}, \partial \Omega) = p^{-\kappa}$;

C4. There exists an $i_0 \in \{1, \ldots, m\}$ such that $|\xi^p_{i_0} - q| = p^{-\kappa}$;

C5. There exist some indices $i_0, k_0, i_0 \neq k_0$ such that $|\xi^p_{i_0} - \xi^p_{k_0}| = p^{-\kappa}$.

From (A3)-(A6), (1.8) and the maximum principle we obtain that for all $i = 1, \ldots, l$ and $k = 1, \ldots, m$ with $i \neq k$,

$$G(q, \xi_i^p) > 0, \quad G(\xi^p_{i_0}, \xi^p_j) > 0, \quad H(\xi_i^p, \xi_i^p) > 0, \quad H(\xi_i^p, \xi_i^p) = -\frac{1}{2\pi} \log \left[ \text{dist}(\xi^p_i, \partial \Omega) \right] + O(1). \quad (5.8)$$

For the first and second cases, by (5.6) and (5.8) we have

$$\max_{\xi \in \partial O_p(q)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ p\left[ (m + l + 2 + 2\alpha)a(q) - d_0 \right] + O(\log p) \right\}, \quad (5.9)$$

which contradicts to (5.7). This shows that $a(\xi^p_i) \to a(q)$. By the condition over $a$, we get $\xi^p_i \to q$ for all $i = 1, \ldots, m$.

For the third case, by (5.6) and (5.8) we have that if $0 < \alpha \in \mathbb{N}^*$,

$$\max_{\xi \in \partial O_p(q)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ \sum_{i=1}^l a(\xi^p_i)[p - 22\log p - 4\alpha a(q) - d_0 - O(1)] + \sum_{i=1}^m a(\xi^p_i)[p - 2 \log p - 4a(\xi^p_i)] + O(1) \right\}$$

$$+ 2(1 + \alpha)a(q) \left\{ p - 2 \log p - 16\pi \sum_{i=1}^l G(q, \xi^p_i) + 8m \sum_{i=1}^l \log |\xi^p_i - q| \right\}$$

$$\leq \frac{2\pi e}{p^2} \left\{ \sum_{i=1}^l a(q)[p - 2 \log p - 4a(\xi^p_i) + O(1)], \ \sum_{i=1}^m a(q)[p - 2 \log p - 4a(\xi^p_i) + O(1)] \right\}$$

$$+ 2(1 + \alpha)a(q) \left\{ p - 2 \log p - 16\pi \sum_{i=1}^l G(q, \xi^p_i) + 8m \sum_{i=1}^l \log |\xi^p_i - q| \right\}$$

$$\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 8\alpha a(\xi^p_i) \log p + O(1) \right\}, \quad (5.10)$$

where the last inequality is due to the fact that for any $i = 1, \ldots, l$, by (A3) and (1.9),

$$-4\alpha \log |\xi^p_i - q| - 16\pi(1 + \alpha)G(q, \xi^p_i) = (8 + 4\alpha) \log |\xi^p_i - q| - 16\pi(1 + \alpha)H(q, \xi^p_i) \leq C,$$

and for any $i = l + 1, \ldots, m$,

$$-4\alpha \log |\xi^p_i - q| + 16(1 + \alpha) \log |\xi^p_i - q| = (16 + 12\alpha) \log |\xi^p_i - q| \leq C.$$
with some large constant \( C > 0 \). While if \(-1 < \alpha < 0\),

\[
\max_{\xi \in \Omega_p(\xi)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 16\pi a(\xi_{i_0}) H(\xi_{i_0}^p, \xi_{k_0}^p) + O(1) \right\}
\]

\[
\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 8\kappa a(\xi_{i_0}) \log p + O(1) \right\}. \tag{5.11}
\]

For the fourth case, by (5.6) and (5.8) we have that if \( i_0 \in \{1, \ldots, l\}, \)

\[
\max_{\xi \in \Omega_p(\xi)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 16\pi a(\xi_{i_0}) H(\xi_{i_0}^p, \xi_{i_0}^p) + O(1) \right\}
\]

\[
\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 8\kappa a(\xi_{i_0}) \log p + O(1) \right\}
\] (5.12)

because \( q \in \partial \Omega \) and \( p^{-\kappa} \leq \text{dist}(\xi_{i_0}^p, \partial \Omega) \leq |\xi_{i_0}^p - q| = p^{-\kappa} \), while if \( i_0 \in \{l + 1, \ldots, m\}, \)

\[
\max_{\xi \in \Omega_p(\xi)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) + (\frac{16}{\pi} + \alpha) a(\xi_{i_0}) \log |\xi_{i_0}^p - q| + O(1) \right\}
\]

\[
\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 4\kappa a(\xi_{i_0}) \log p + O(1) \right\}. \tag{5.13}
\]

For the last case, by (5.6) and (5.8) we have that if \( i_0 \in \{1, \ldots, m\} \) and \( k_0 \in \{1, \ldots, l\}, \)

\[
\max_{\xi \in \Omega_p(\xi)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) + 8a(\xi_{i_0}) \log |\xi_{i_0}^p - \xi_{k_0}^p| + O(1) \right\}
\]

\[
\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 8\kappa a(\xi_{i_0}) \log p + O(1) \right\}, \tag{5.14}
\]

while if \( i_0 \in \{l + 1, \ldots, m\} \) and \( k_0 \in \{l + 1, \ldots, m\}, \)

\[
\max_{\xi \in \Omega_p(\xi)} F_p(\xi) \leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) + 4a(\xi_{i_0}) \log |\xi_{i_0}^p - \xi_{k_0}^p| + O(1) \right\}
\]

\[
\leq \frac{2\pi e}{p^2} \left\{ (m + l + 2 + 2\alpha)(p - 2 \log p) a(q) - 4\kappa a(\xi_{i_0}) \log p + O(1) \right\}. \tag{5.15}
\]

Comparing (5.10)-(5.15) with (5.7), we obtain

\[
2(m + l + 2 + 2\alpha)a(q) \log p + 8\kappa a(\xi_{i_0}^p) \log p \leq 2(m + l)^2 + (8 + 6\alpha)(m + l) + 4 + 4\alpha]a(q) \log p + O(1), \tag{5.16}
\]

which is impossible by the choice of \( \kappa \) in (2.4).

\[\square\]

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**Appendix A**

In this appendix we list some properties of the regular part of Green’s function and its corresponding Robin’s function, see [2] for proofs.

Let the vector function \( T(x) = (T_1(x), T_2(x)) \) be the solution of

\[
\Delta_x T - T = \frac{x}{|x|^2} \quad \text{in} \quad \mathbb{R}^2, \quad T(x) \in L^\infty_{\text{loc}}(\mathbb{R}^2). \tag{A1}
\]

Standard elliptic regularity theory yields that \( T(x) \in W^{2,\sigma}_{\text{loc}}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{0\}) \) for any \( 1 < \sigma < 2 \), and further the Sobolev embeddings imply that \( T(x) \in W^{1,1/\beta}(B_r(0)) \cap C^\beta(B_r(0)) \) for any \( r > 0 \) and \( 0 < \beta < 1 \).
**Lemma A.1.** Let $T(x)$ be the function described in (A1). There exists a function $H_1(x, y)$ such that

(i) for every $x, y \in \Omega$,

\[
H(x, y) = H_1(x, y) + \begin{cases} 
\frac{1}{2\pi} \nabla \log a(y) \cdot T(x - y), & y \in \Omega, \\
\frac{1}{\pi} \nabla \log a(y) \cdot T(x - y), & y \in \partial \Omega,
\end{cases} \tag{A2}
\]

(ii) the mapping $y \in \Omega \mapsto H_1(\cdot, y)$ belongs to $C^1(\Omega, C^1(\overline{\Omega})) \cap C^1(\partial \Omega, C^1(\overline{\Omega}))$.

In this way, $y \in \Omega \mapsto H(\cdot, y) \in C^{\beta}(\Omega, C^2(\overline{\Omega})) \cap C(\partial \Omega, C^3(\overline{\Omega}))$ and $H(x, y) \in C^2(\overline{\Omega} \times \Omega) \cap C^1(\Omega \times \partial \Omega)$ for any $\beta \in (0, 1)$, and the corresponding Robin’s function $y \in \Omega \mapsto H(y, y)$ belongs to $C^1(\Omega) \cap C^1(\partial \Omega)$.

Let $\Omega_d := \{ y \in \Omega \mid \text{dist}(y, \partial \Omega) < d \}$ with $d > 0$ sufficiently small but fixed. Then for any $y \in \Omega_d$, there exists a unique reflection of $y$ across $\partial \Omega$ along the outer normal direction, $y^* \in \Omega^*$, such that $|y - y^*| = 2 \text{dist}(y, \partial \Omega)$.

**Lemma A.2.** There exists a mapping $y \in \Omega_d \mapsto z(\cdot, y) \in C(\Omega_d, C^\beta(\overline{\Omega})) \cap L^\infty(\Omega_d, C^{\beta}(\overline{\Omega}))$ for any $\beta \in (0, 1)$ such that for any $x \in \overline{\Omega}$ and $y \in \Omega_d$,

\[
H(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y^*|} + z(x, y), \tag{A3}
\]

and

\[
z(x, y) = \frac{1}{2\pi} \nabla \log a(y) \cdot T(x - y) - \frac{1}{2\pi} \nabla \log a(y^*) \cdot T(x - y^*) + \tilde{z}(x, y), \tag{A4}
\]

where the mapping $y \in \Omega_d \mapsto \tilde{z}(\cdot, y)$ belongs to $C^1(\overline{\Omega}_d, C^1(\overline{\Omega}))$.

**Lemma A.3.** The Robin’s function $y \in \Omega \mapsto H(y, y)$ satisfies

\[
H(y, y) = \frac{1}{2\pi} \log \frac{1}{|y - y^*|} + z(y), \quad \forall \ y \in \Omega_d, \tag{A5}
\]

where $z \in C^1(\overline{\Omega}_d)$ and

\[
z(y) = \frac{1}{2\pi} \nabla \log a(y) \cdot T(0) - \frac{1}{2\pi} \nabla \log a(y^*) \cdot T(y - y^*) + \tilde{z}(y, y), \quad \forall \ y \in \Omega_d. \tag{A6}
\]
Proof of Lemma 2.1. Observe first that, for any $0 < \tau < 1$, by (2.1), (2.6) and (2.12),

\[
- \Delta_{\alpha} H_0 + H_0 = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left\{ \left[-4(1 + \alpha) + \frac{C_1}{p} + \frac{C_2}{p^2} \right] \left[ |x - q|^{2\alpha} (x - q) \cdot \nabla \log a(x) - \frac{\log (\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)})}{2(1 + \alpha)} \right] 
- \log \left[ 8(1 + \alpha)^2 \delta_0^{2(1+\alpha)} \right] + \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \delta_0 
\right. \\
+ \frac{1}{p} O \left( \partial_{\Omega \setminus B_{\delta_0^{2(1+\alpha)}}(q)} \right) \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{1+\alpha} + |x - q|^{1+\alpha}} + \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \\
+ \frac{1}{p} O \left( \partial_{\Omega \cap B_{\delta_0^{2(1+\alpha)}}(q)} \right) \left| \frac{|x - q|^{2\alpha}}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} \right| \left( |x - q|^{2(1+\alpha)} \cdot \nabla \log a(x) \right) \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \right\} \text{ in } \Omega,
\]

\[
\frac{\partial H_0}{\partial \nu} = \frac{1}{\gamma \mu_0^{2/(p-1)}} \left\{ \left[-4(1 + \alpha) + \frac{C_1}{p} + \frac{C_2}{p^2} \right] \left[ |x - q|^{2\alpha} (x - q) \cdot \nu(x) + \frac{1}{p} O \left( \partial_{\Omega \setminus B_{\delta_0^{2(1+\alpha)}}(q)} \right) \right] \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{1+\alpha} + |x - q|^{1+\alpha}} + \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \right\} \text{ on } \partial \Omega.
\]

Observing that $q \in \overline{\Omega}$, by (1.8), (1.9) and (2.20) we find that the regular part of Green’s function, $H(x, q)$, satisfies

\[
- \Delta_{\alpha} H(x, q) + H(x, q) = \frac{4(1 + \alpha)}{c_0} \log |x - q| - \frac{4(1 + \alpha)}{c_0} \frac{(x - q) \cdot \nabla \log a(x)}{|x - q|^2} \text{ in } \Omega,
\]

\[
\frac{\partial H(x, q)}{\partial \nu} = \frac{4(1 + \alpha)}{c_0} \frac{(x - q) \cdot \nu(x)}{|x - q|^2} \text{ on } \partial \Omega.
\]

Thus if we define

\[
Z_0(x) = \gamma \mu_0^{2/(p-1)} H_0(x) - \left( 1 - \frac{C_1}{4(1+\alpha)p} - \frac{C_2}{4(1+\alpha)p^2} \right) c_0 H(x, q) + \log \left( 8(1 + \alpha)^2 \delta_0^{2(1+\alpha)} \right) - \left( \frac{C_1}{p} + \frac{C_2}{p^2} \right) \log \delta_0,
\]

then $Z_0(x)$ satisfies

\[
- \Delta_{\alpha} Z_0 + Z_0 = \left[ -4(1 + \alpha) + \frac{C_1}{p} + \frac{C_2}{p^2} \right] \left[ \frac{1}{2(1 + \alpha)} \log \frac{|x - q|^{2(1+\alpha)}}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} - \frac{\delta_0^{2(1+\alpha)} (x - q) \cdot \nabla \log a(x)}{|x - q|^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} \right]
\]

\[
+ \frac{1}{p} O \left( \partial_{\Omega \setminus B_{\delta_0^{2(1+\alpha)}}(q)} \right) \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{1+\alpha} + |x - q|^{1+\alpha}} + \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \\
+ \frac{1}{p} O \left( \partial_{\Omega \cap B_{\delta_0^{2(1+\alpha)}}(q)} \right) \left| \frac{|x - q|^{2\alpha}}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} \right| \left( |x - q|^{2(1+\alpha)} \cdot \nabla \log a(x) \right) \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \right\} \text{ in } \Omega,
\]

\[
\frac{\partial Z_0}{\partial \nu} = \left[ -4(1 + \alpha) + \frac{C_1}{p} + \frac{C_2}{p^2} \right] \left( \frac{(x - q) \cdot \nu(x)}{|x - q|^2} \delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)} \right) + \frac{1}{p} O \left( \partial_{\Omega \setminus B_{\delta_0^{2(1+\alpha)}}(q)} \right) \left( \frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \right) \right\} \text{ on } \partial \Omega.
\]
Applying the polar coordinates with center \( q \), i.e. \( r = |x - q| \), and using the change of variables \( s = r/\delta_0 \), we get that for any \( \theta > 1 \),

\[
\int_{\Omega} \left| \log \frac{|x - q|^{2(1 + \alpha)}}{\delta_0^{2(1 + \alpha)} + |x - q|^{2(1 + \alpha)}} \right|^\theta dx \leq C \int_{0}^{\text{diam}(\Omega) \delta_0} \left| \log \frac{r^{2(1 + \alpha)}}{\delta_0^{2(1 + \alpha)} + r^{2(1 + \alpha)}} \right|^\theta rdr
\]

\[
\leq C\delta_0^2 \int_{0}^{\text{diam}(\Omega) / \delta_0} \left| \log \left( 1 + \frac{1}{s^{2(1 + \alpha)}} \right) \right|^\theta ds
\]

\[
\leq C \left[ \delta_0^2 + \delta_0^{2\theta(1 + \alpha)} \right],
\]

and

\[
\int_{\Omega \cap B_{\delta_0^{\tau/2}}(q)} \left| \frac{\delta_0^{1 + \alpha}}{\delta_0^{1 + \alpha} + |x - q|^{1 + \alpha}} + \frac{\delta_0^{1 + \alpha}}{\delta_0^{2 + \alpha} + |x - q|^{2 + \alpha}} \right|^\theta dx \leq C \int_{\delta_0^{\tau/2}}^{\text{diam}(\Omega) / \delta_0} \left[ \frac{\delta_0^{\theta(1 + \alpha)}}{(\delta_0^{1 + \alpha} + r^{1 + \alpha})^{\theta}} + \frac{\delta_0^{\theta(1 + \alpha)}}{(\delta_0^{2 + \alpha} + r^{2 + \alpha})^{\theta}} \right] rdr
\]

\[
\leq C \int_{\delta_0^{\tau/2}}^{\text{diam}(\Omega) / \delta_0} \left[ \frac{\delta_0^{2}}{(1 + s^{1 + \alpha})^{\theta}} + \frac{s^{2 \theta - \theta}}{(1 + s^{2(1 + \alpha)})^{\theta}} \right] ds
\]

\[
\leq C \left[ \delta_0^{\theta(1 + \alpha)} + \delta_0^{\tau + \theta(1 + \alpha) - \frac{\tau}{2} \theta(2 + \alpha)} \right],
\]

and for any \( 1 < \theta < 2 \),

\[
\int_{\Omega \cap B_{\delta_0^{\tau/2}}(q)} \left| \frac{|x - q|^{2\alpha}}{\delta_0^{2(1 + \alpha)} + |x - q|^{2(1 + \alpha)}} \right|^\theta dx \leq C \int_{0}^{\delta_0^{\tau/2}} \left[ \frac{r^{1 + 2\alpha}}{\delta_0^{2(1 + \alpha)} + r^{2(1 + \alpha)}} \right]^\theta rdr
\]

\[
\leq C \int_{0}^{\delta_0^{\tau/2}} \left[ \frac{s^{1 + 2\alpha}}{1 + s^{2(1 + \alpha)}} \right]^\theta ds
\]

\[
\leq C \delta_0^{\tau(2 - \theta)/2},
\]

and

\[
\int_{\Omega} \left| \frac{(x - q) \cdot \nabla \log a(x)}{|x - q|^2} \delta_0^{2(1 + \alpha)} \right|^\theta dx \leq C \int_{0}^{\text{diam}(\Omega)} \left| \frac{\delta_0^{2(1 + \alpha)}}{r \left( \delta_0^{2(1 + \alpha)} + r^{2(1 + \alpha)} \right)} \right|^\theta rdr
\]

\[
\leq C \delta_0^{2 - \theta} \int_{0}^{\text{diam}(\Omega) / \delta_0} \frac{s^{1 - \theta}}{(1 + s^{2(1 + \alpha)})^{\theta}} ds
\]

\[
\leq C \left( \delta_0^{2 - \theta} + \delta_0^{2\theta(1 + \alpha)} \right).
\]

Then for any \( 1 < \theta < 2 \),

\[
\| - \Delta a Z_0 + Z_0 \|_{L^p(\Omega)} \leq C \delta_0^{\tau(1/\theta-1/2)}. \quad (B1)
\]
On the other hand, if \( q \in \partial \Omega \), then, by using the fact that \( |(x - q) \cdot \nu(q)| \leq C|x - q|^2 \) for any \( x \in \partial \Omega \) (see [2]), we have that for any \( \theta > 1 \),

\[
\int_{\partial \Omega} \frac{|x - q|^{2(1+\alpha)}}{|x - q|^2} \frac{\delta_0^{2(1+\alpha)} \delta_0}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} dx \leq C \int_{0}^{\|\partial \Omega\|} \frac{\delta_0^{2(1+\alpha)}}{\left(\delta_0^{2(1+\alpha)} + r^{2(1+\alpha)}\right)^\theta} dr
\]

\[
= C \delta_0 \int_{0}^{\|\partial \Omega\|/\delta_0} \frac{1}{(1 + s^{2(1+\alpha)})^\theta} ds \leq C \left( \delta_0 + \delta_0^{2(1+\alpha)} \right),
\]

and

\[
\int_{\partial \Omega \cap B_{\delta_0^{\tau/2}}(q)} \frac{|x - q|^{2\alpha}|(x - q) \cdot \nu(x)|}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} dx \leq C \int_{\partial \Omega \cap B_{\delta_0^{\tau/2}}(q)} \frac{|x - q|^{2(1+\alpha)}}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} dx
\]

\[
\leq C \left| \partial \Omega \cap B_{\delta_0^{\tau/2}}(0) \right| \leq C \delta_0^{\tau/2},
\]

and

\[
\int_{\partial \Omega \cap B_{\delta_0^{\tau/2}}(q)} \frac{\delta_0^{1+\alpha} \delta_0^{2+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} dx \leq C \int_{\delta_0^{\tau/2}} \left( \frac{\delta_0^{1+\alpha} \delta_0^{2+\alpha}}{\delta_0^{2+\alpha} + r^{2+\alpha}} \right)^\theta dr = C \int_{\delta_0^{\tau/2-1}} \frac{\delta_0^{1-\theta}}{(1 + s^{2+\alpha})^\theta} ds
\]

\[
\leq C \int_{\delta_0^{\tau/2-1}} \frac{\delta_0^{1-\theta}}{s^{\theta(2+\alpha)}} ds \leq C \delta_0^{\theta(1+\alpha) + \frac{\theta}{2} - \frac{\theta}{2} \tau^2(2+\alpha)},
\]

which implies that for any \( q \in \partial \Omega \), \( \theta > 1 \) and \( 0 < \tau < \min\{1, 2(1 + \alpha)/(2 + \alpha)\} \),

\[
\left\| \frac{\partial Z_0}{\partial \nu} \right\|_{L^\theta(\partial \Omega)} \leq C \delta_0^{\tau/2\theta}.
\]  \(^{(B2)}\)

While if \( q \in \Omega \), then we estimate that for any \( x \in \partial \Omega \),

\[
\frac{|(x - q) \cdot \nu(x)|}{|x - q|^2} \frac{\delta_0^{2(1+\alpha)}}{\delta_0^{2(1+\alpha)} + |x - q|^{2(1+\alpha)}} \leq \frac{\delta_0^{2(1+\alpha)}}{|x - q|^{3+2\alpha}} \leq C \delta_0^{2(1+\alpha)},
\]

\[
\frac{\delta_0^{1+\alpha}}{\delta_0^{2+\alpha} + |x - q|^{2+\alpha}} \leq \frac{\delta_0^{1+\alpha}}{|x - q|^{2+\alpha}} \leq C \delta_0^{1+\alpha},
\]

and then,

\[
\left\| \frac{\partial Z_0}{\partial \nu} \right\|_{L^\infty(\partial \Omega)} \leq C \delta_0^{1+\alpha}.
\]  \(^{(B3)}\)

As a consequence, by (B1)-(B3) and elliptic regularity theory we conclude that for any \( 1 < \theta < 2 \), \( 0 < \tau < \min\{1, 2(1 + \alpha)/(2 + \alpha)\} \) and \( 0 < \lambda < 1/\theta \),

\[
\left\| Z_0 \right\|_{W^{1+\lambda, \theta}(\Omega)} \leq C \left( \left\| \Delta_a Z_0 + Z_0 \right\|_{L^\theta(\Omega)} + \left\| \frac{\partial Z_0}{\partial \nu} \right\|_{L^\theta(\partial \Omega)} \right) \leq C \delta_0^{\tau(1/\theta - 1/2)}.
\]

Furthermore, by Morrey’s embedding theorem we have that for any \( 0 < \sigma < 1/2 + 1/\theta \),

\[
\left\| Z_0 \right\|_{C^{\sigma}(\overline{\Omega})} \leq C \delta_0^{\tau(1/\theta - 1/2)},
\]

which implies that expansion \((2.21)\) holds with \( \beta = 2\tau(1/\theta - 1/2) \). In addition, expansion \((2.22)\) can be also derived from these analogous arguments of \((2.21)\).
Proof of Lemma 2.2. Making the change of variables \( s = 1/p \), we have that relations (2.25) and (2.27) are equivalent to the homogeneous equations \((S_0(s, \xi, \mu), \ldots, S_m(s, \xi, \mu)) = (0, \ldots, 0)\), namely

\[
S_0(s, \xi, \mu) := \log \mu_0 - \frac{1}{4}c_0H(q, q) + \frac{\log 8(1 + \alpha)^2}{4 - \frac{1}{1+\alpha}C_1s - \frac{1}{1+\alpha}C_2s^2} + \frac{1}{4}C_1s - \frac{1}{4}C_2s^2 \sum_{k=1}^{m} \left( \frac{\mu_0}{\mu_k |\xi_k - q|^\alpha} \right)^{2s/(1-s)} c_k G(q, \xi_k) = 0,
\]
and for each \( i = 1, \ldots, m \),
\[
S_i(s, \xi, \mu) := \log \mu_i - \frac{1}{4}c_i H(\xi_i, \xi_i) + \frac{\log 8 + \frac{1}{4}C_1 + \frac{1}{4}C_2s}{4 - C_1s - C_2s^2} - \frac{1}{4} \sum_{k=1, k \neq i}^{m} \left( \frac{\mu_i |\xi_i - q|^\alpha}{\mu_k |\xi_k - q|^\alpha} \right)^{2s/(1-s)} c_k G(\xi_i, \xi_k) = 0.
\]

Obviously, from the definitions of the constants \( C_1 \) and \( \widetilde{C}_1 \) in (2.15) we deduce that for \( s = 0 \),
\[
\mu_0(0, \xi) = e^{-\frac{1}{4}c_0H(q, q) + \frac{1}{4}c_0 G(q, \xi)}, \quad (B4)
\]
and for each \( i = 1, \ldots, m \),
\[
\mu_i(0, \xi) = e^{-\frac{1}{4}c_i H(\xi_i, \xi_i) + \frac{1}{4}c_0 G(q, \xi) + \frac{1}{4}c_i G(\xi_i, \xi_i)}, \quad (B5)
\]

On the other hand, observe that for any \( s > 0 \) small enough,
\[
\left( \frac{C^2}{s^{C+\kappa}} \right)^{2s/(1-s)} = e^\frac{2s}{1-s} \left[ 2 \log C + (C+\kappa) \log \frac{1}{s} \right] = 1 + O \left( s \log \frac{1}{s} \right).
\]

Then by (2.3) and the first estimate of (2.28) we find that for any \( i, k = 1, \ldots, m \) and \( i \neq k \),
\[
\left( \frac{\mu_0}{\mu_k |\xi_k - q|^\alpha} \right)^{2s/(1-s)} = 1 + O \left( s \log \frac{1}{s} \right) \quad \text{and} \quad \left( \frac{\mu_i |\xi_i - q|^\alpha}{\mu_k |\xi_k - q|^\alpha} \right)^{2s/(1-s)} = 1 + O \left( s \log \frac{1}{s} \right). \quad (B6)
\]

Moreover, by (A3), (1.9), (2.3) and the fact that \( a(\xi_i)G(\xi_i, \xi_k) = a(\xi_k)G(\xi_k, \xi_i) \) and \( a(\xi_i)G(\xi_i, q) = a(q)G(q, \xi_i) \) for all \( i, k = 1, \ldots, m \) with \( i \neq k \), we can conclude that for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_{1/s}(q) \),
\[
G(\xi_i, \xi_k) = O \left( \log \frac{1}{s} \right) \quad \text{and} \quad G(\xi, \xi_i) = O \left( \log \frac{1}{s} \right), \quad \forall \ i, k = 1, \ldots, m, \ i \neq k. \quad (B7)
\]

Furthermore, some direct computations easily deduce that for each \( i, k = 1, \ldots, m \) with \( i \neq k \),
\[
\frac{\partial S_0(s, \xi, \mu)}{\partial \mu_0} = \frac{1}{\mu_0} \left[ 1 + O \left( s \log \frac{1}{s} \right) \right], \quad \frac{\partial S_0(s, \xi, \mu)}{\partial \mu_k} = \frac{1}{\mu_k} O \left( s \log \frac{1}{s} \right),
\]
and
\[
\frac{\partial S_i(s, \xi, \mu)}{\partial \mu_k} = \frac{1}{\mu_k} \left[ 1 + O \left( s \log \frac{1}{s} \right) \right], \quad \frac{\partial S_i(s, \xi, \mu)}{\partial \mu_0} = \frac{1}{\mu_0} O \left( s \log \frac{1}{s} \right), \quad \frac{\partial S_i(s, \xi, \mu)}{\partial \mu_k} = \frac{1}{\mu_k} O \left( s \log \frac{1}{s} \right).
\]

If we set \( S(s, \xi, \mu) = (S_0(s, \xi, \mu), \ldots, S_m(s, \xi, \mu)) \), then the vector function \( S(s, \xi, \mu) \) satisfies
\[
\det \left( \nabla_\mu S(s, \xi, \mu) \right) = \frac{1}{\mu_0 \mu_1 \cdots \mu_m} \left[ 1 + O \left( s \log \frac{1}{s} \right) \right] \neq 0.
\]

Consequently, using the implicit function theorem, we can derive that the homogeneous equations \( S(s, \xi, \mu) = (0, \ldots, 0) \) is solvable in some neighborhood of \((0, \xi, \mu(0, \xi))\), namely for any points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_{p}(q) \) and
any $p > 1$ large enough, systems (2.25) and (2.27) have a unique solution $\mu = (\mu_0, \mu_1, \ldots, \mu_m)$ satisfying the first estimate in (2.28). From (B4)-(B7) it follows that

$$\mu_0 = \mu_0(p, \xi) \equiv e^{-\frac{3}{4} + \frac{1}{4} c_0 H(q, q) + \frac{1}{4} \sum_{k=1}^{m} c_k G(q, \xi_k)} \left[ 1 + O \left( \frac{\log^2 p}{p} \right) \right],$$

and for each $i = 1, \ldots, m$,

$$\mu_i = \mu_i(p, \xi) \equiv e^{-\frac{3}{4} + \frac{1}{4} c_i H(\xi, \xi_i) + \frac{1}{4} c_0 G(\xi, q) + \frac{1}{4} \sum_{k=1, k \neq i}^{m} c_k G(\xi, \xi_k)} \left[ 1 + O \left( \frac{\log^2 p}{p} \right) \right].$$

Moreover, by (A3), (A5), (B6), (B7), (2.3), (2.25) and (2.27) we find that the second estimate in (2.28) holds. □

References


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